Section 13.5. Uniform continuity

Definition. Let $S$ be a non-empty subset of $\mathbb{R}$. We say that a function $f : S \to \mathbb{R}$ is uniformly continuous on $S$ if, for each $\epsilon > 0$, there is a real number $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in S$ with $|x - y| < \delta$, where $\delta$ depends only on $\epsilon$.

Remarks.

a) Uniform continuity is a property involving a function $f$ and a set $S$ on which it is defined. It makes no sense to speak of $f$ being uniformly continuous at a point of $S$ (except when $S$ consists of a single point!).

b) Remember that the $\delta$ in the usual definition of ordinary continuity depends both on $\epsilon$ and the point $x_0$ at which continuity is being tested. So, uniform continuity implies continuity. But the converse is false as we will see below (in Example 5.)

Example 1. Let $f(x) = 5x + 8$. Prove that $f$ is uniformly continuous on $\mathbb{R}$.

Solution We have:

$$|f(x) - f(y)| = 5|x - y|$$

Given $\epsilon > 0$, we choose $\delta = \epsilon/5$ then $|f(x) - f(y)| < \epsilon$ for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Therefore $f$ is uniformly continuous on $\mathbb{R}$.

Example 2. Let $f(x) = x^2$ on $I = [0, 1]$. Prove that $f$ is uniformly continuous on $I$.

Solution We have:

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \leq 2|x - y| \text{ for } x, y \text{ in } [0, 1].$$

Given $\epsilon > 0$, we choose $\delta = \epsilon/2$ then $|f(x) - f(y)| < \epsilon$ for all $x, y \in [0, 1]$ with $|x - y| < \delta$. Therefore $f$ is uniformly continuous on $I = [0, 1]$.

Example 3. Let

$$f(x) = \frac{1}{x^2} \text{ defined on } x \text{ in } I = [3, \infty).$$
Show that \( f \) is uniformly continuous on \( I = [3, \infty) \).

**Solution** We have:

\[
|f(x) - f(y)| = \left| \frac{x^2 - y^2}{x^2y^2} \right| = \left| \frac{(x+y)(x-y)}{x^2y^2} \right| = \left( \frac{1}{xy^2} + \frac{1}{x^2y} \right) |x-y| \leq \frac{2}{27} |x-y|.
\]

Given \( \epsilon > 0 \), we choose \( \delta = \frac{27\epsilon}{2} \) then \( |f(x) - f(y)| < \epsilon \) for all \( x, y \in [3, \infty) \) with \( |x-y| < \delta \). Therefore \( f \) is uniformly continuous on \( [3, \infty) \).

**Example 4.** Let \( f(x) = \cos x \). Prove that \( f \) is uniformly continuous on \( \mathbb{R} \).

**Solution.** We have seen that

\[
|\cos x - \cos y| \leq |x - y|, \quad \text{for all } x, y \in \mathbb{R}
\]

Given \( \epsilon > 0 \), we choose \( \delta = \epsilon \) then \( |f(x) - f(y)| < \epsilon \) for all \( x, y \in I \) with \( |x-y| < \delta \). Therefore \( f \) is uniformly continuous on \( \mathbb{R} \).

**Example 5.** Let

\[ f(x) = \frac{1}{x} \]

defined on \( I = (0, 1) \).

Show that \( f \) is not uniformly continuous on \( I \).

**Solution** The function \( f(x) = \frac{1}{x} \) is certainly continuous on the open interval \((0, 1)\). But it is not uniformly continuous on \((0, 1)\). Indeed, let \( x \) and \( y \) be elements of \((0, 1)\) then

\[
|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{y-x}{xy} \right|.
\]

Here, the key idea is the fact that \( |x-y|/xy \) might approach infinity if \( x \) and \( y \) approach zero, and therefore \( |f(x) - f(y)| \) will be bigger than any chosen \( \epsilon \).

Now, we will formalize this idea. We pick \( \epsilon = 1 \). Without loss of generality, we can suppose that \( 0 < \delta < 1 \). Set \( x = \frac{\delta}{2} \) and \( y = \delta \) then

\[
|f(x) - f(y)| = \frac{2}{\delta} - \frac{1}{\delta} = \frac{1}{\delta} > \epsilon.
\]

Therefore, the function is not uniformly continuous on \((0, 1)\).

**Theorem.** Let \( I \) be a compact interval. Then \( f : I \to \mathbb{R} \) is continuous on \( I \) if and only if \( f \) is uniformly continuous.