Chapter 1 - Overview

Section 2.1: The Real Numbers
• Axioms of a field
• Ordered sets - Ordered fields
• Bounded sets

Section 2.2: Infinite Decimals
• Infinite decimal expansions
• Real numbers as infinite decimals

Section 2.3: Limits
• N-epsilon definition of limits
• Comparison theorem for sequences
• The squeeze theorem for sequences

Section 2.4: Basic Properties of Limits

Section 2.5: Upper and Lower Bounds
• Supremum and infimum
• Least upper bound principle
• Monotone convergence theorem
• Limsups and liminfs

Section 2.6: Subsequences
• Nested intervals lemma
• Bolzano-Weierstrass Theorem

Section 2.7: Cauchy Sequences
• Complete sets
• Completeness theorem for \( \mathbb{R} \)

Section 2.8: Cardinality
• Injections, surjections, bijections
• Countable and uncountable sets - Cardinals
Chapter 3 - Overview

Section 3.1: Convergent Series

• Definition of a convergent series
• Basic examples: the harmonic series, telescoping sums, geometric series.
• Cauchy criterion for series

Section 3.2: Convergence Test for Series

• Series with positive terms - Bounded sum test

A series \( \sum_{n=1}^{\infty} a_n \) with positive terms converges if and only if the sequence \( (s_n)_{n=1}^{\infty} \) of partial sums is bounded.

• Divergence test

Let \( (a_n)_{n=1}^{\infty} \) be a sequence of real numbers. If the sequence \( (a_n)_{n=1}^{\infty} \) does not converge to zero, then the series \( \sum_{n=1}^{\infty} a_n \) diverges.

• The comparison test for series

Let \( (a_n)_{n=1}^{\infty} \) and \( (b_n)_{n=1}^{\infty} \) be two series of real numbers with \( |a_n| \leq b_n \) for all \( n \in \mathbb{N} \). If \( \sum_{n=1}^{\infty} b_n \) converges then \( \sum_{n=1}^{\infty} a_n \) converges absolutely.

Example: Show that \( \sum_{n=1}^{\infty} 2^{-n} \cos n \) converges.

• Alternating series test

Suppose \( (a_n)_{n=1}^{\infty} \) is a monotone decreasing sequence of real numbers that converges to zero, and \( a_n \geq 0 \), for all \( n \). Then, the alternating series \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges.

Example: Show that \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges.

• The root test

Let \( (a_n)_{n=1}^{\infty} \) be a sequence of real numbers. Consider the series \( \sum_{n=1}^{\infty} a_n \).

Then:

- if \( \lim \sup \sqrt[n]{|a_n|} < 1 \) then the series converges absolutely
- if \( \lim \sup \sqrt[n]{|a_n|} > 1 \) then the series diverges
- if \( \lim \sup \sqrt[n]{|a_n|} = 1 \), this test gives no information

Example: Show that \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} \) converges.
• **The ratio test**

Let \((a_n)_{n=1}^{\infty}\) be a sequence of real numbers. Suppose that

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \varrho.
\]

If \(\varrho < 1\) then the series \(\sum_{n=1}^{\infty} a_n\) converges.
If \(\varrho > 1\) or if \(\varrho = \infty\), then the series \(\sum_{n=1}^{\infty} a_n\) diverges.
If \(\varrho = 1\) this test gives is inconclusive.

Example 1: Show that Euler’s series \(\sum_{n=1}^{\infty} \frac{1}{n!}\) converges.
Example 2: Show that the series \(\sum_{n=1}^{\infty} \frac{2^n}{n!}\) converges.

• **The integral test**

Suppose that \(f(x)\) is a positive, continuous, and monotone decreasing function on the interval \([1, \infty)\). Let \(a_n = f(n)\), for \(n \in \mathbb{N}\). Then \(\sum_{n=1}^{\infty} a_n\) converges if and only if the integral \(\int_1^{\infty} f(x)dx\) converges.

Example: Show that \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) converges if \(p > 1\).

Section 3.4: Absolute and Conditional Convergence

• Absolute Convergence

If \(\sum_{n=1}^{\infty} |a_n|\) converges, then the series \(\sum_{n=1}^{\infty} a_n\) also converges. The converse is false.

• **Rearrangement theorem**

Suppose \(\sum_{n=1}^{\infty} a_n\) is *conditionally convergent*. Then, for any real number \(\ell\), there is a rearrangement of the series that converges to \(\ell\).

**Chapter 4 - Overview**

Section 4.1: The Euclidean space \(\mathbb{R}^n\)

• Schwarz inequality
• Triangle Inequality
• Orthonormal sets

Section 4.2: Convergence and completeness in \(\mathbb{R}^n\)

• Cauchy sequences in \(\mathbb{R}^n\)
• *Completeness theorem for \(\mathbb{R}^n\)*

Section 4.3: Closed and open subsets of \(\mathbb{R}^n\)

Section 4.4: Compact Sets and The Heine-Borel Theorem

• Compact Sets - *The Heine-Borel Theorem*
• Cantor’s intersection theorem

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