Problem 1 (16pts). The number \( y \) of absentees (in hundred) in a factory during its first three months is given by

\[
\begin{align*}
y &= 2 \quad \text{at} \quad t = 1; \quad y = 2 \quad \text{at} \quad t = 2; \quad y = 4 \quad \text{at} \quad t = 3;
\end{align*}
\]

Find an equation of the least squares approximating line for the data and give a sketch. Find the least squares error.

Solution Let \( y = c + dt \) be the equation of the line we seek. Then, substituting the three points into the equation, we get:

\[
\begin{align*}
c + d &= 2 \\
c + 2d &= 2 \\
c + 3d &= 4
\end{align*}
\]

or

\[
\begin{bmatrix}
1 & 1 \\
1 & 2 \\
1 & 3
\end{bmatrix}
\begin{bmatrix}
c \\
d
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
2 \\
4
\end{bmatrix}
\]

We denote

\[
A = 
\begin{bmatrix}
1 & 1 \\
1 & 2 \\
1 & 3
\end{bmatrix}
\quad \text{and} \quad 
b = 
\begin{bmatrix}
2 \\
2 \\
4
\end{bmatrix}
\]

Then, the least squares solution is

\[
\mathbf{x} = \left( A^T A \right)^{-1} A^T b = 
\begin{bmatrix}
2/3 \\
1/3 \\
1/3
\end{bmatrix}
\]

Therefore, the equation of the least squares approximating line is

\[
y = \frac{2}{3} + t.
\]

Since, the error vector is

\[
\mathbf{e} = b - Ax = 
\begin{bmatrix}
1/3 \\
-2/3 \\
1/3
\end{bmatrix}
\]

the least squares error is

\[
\| \mathbf{e} \| = \sqrt{2/3}.
\]
Problem 2 (10pts). Use Cramer’s rule to solve the system

\[
\begin{align*}
2x + 3y &= 1 \\
x + 3y &= 5
\end{align*}
\]

Solution

\[
x = \frac{1}{2} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 3 \\ 1 \\ 3 \end{bmatrix} = -4 \quad \text{and} \quad y = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \\ 3 \\ 1 \end{bmatrix} = 3
\]

Problem 3 (16pts).

3.1 Find a basis for the plane \( x - y + 3z = 0 \).

Solution

\[
x - y + 3z = 0 \quad \text{if and only if} \quad x = y - 3z.
\]

The solution set of is equation is

\[
X = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}. \text{So, a basis for the plane is } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]

3.2 Find the 3 by 3 matrix that projects every point in \( \mathbb{R}^3 \) onto the plane \( x - y + 3z = 0 \).

Solution

The projection matrix is

\[
P = A(A^T A)^{-1} A^T = \frac{1}{11} \begin{bmatrix} 10 & 1 & -3 \\ 1 & 10 & 3 \\ -3 & 3 & 2 \end{bmatrix}
\]

Problem 4 (16pts). Let \( \mathbf{a} \) and \( \mathbf{b} \) be the points in \( \mathbb{R}^3 \) given by \( \mathbf{a} = (1, -4, 2) \) and \( \mathbf{b} = (1, 1, 1) \). Find the closest point to \( \mathbf{b} \) on the line through the origin and \( \mathbf{a} \).

Solution

The closest point to \( \mathbf{b} \) on the line through the origin and \( \mathbf{a} \) is

\[
p = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} \mathbf{a} = \frac{-1}{21} \begin{pmatrix} 1, -4, 2 \end{pmatrix} = \left( \frac{-1}{21}, \frac{4}{21}, \frac{-2}{21} \right).
\]
Problem 5 (14 pts).

a. Find the volume of the parallelepiped with four of its vertices at (0, 0, 0), (1, 4, 8), (1, 2, 6), (1, 4, -4).

b. Where are the other four vertices?

Solution By Gaussian elimination, one gets

\[
\begin{vmatrix}
1 & 4 & 8 \\
1 & 2 & 6 \\
1 & 4 & -4
\end{vmatrix}
= \begin{vmatrix}
1 & 4 & 8 \\
0 & -2 & -2 \\
0 & 0 & -12
\end{vmatrix}
= 24
\]

The volume of the parallelepiped equals 24 units\(^3\). The other four vertices are: (2, 6, 14), (2, 6, 2), (2, 8, 4), (3, 10, 10).

Problem 6 (16 pts). Consider the matrix

\[
A = \begin{bmatrix}
1 & 1 \\
1 & 0 \\
1 & 0 \\
1 & 1
\end{bmatrix}
\]

Factor \(A\) into \(A = QR\), where \(Q\) has orthonormal columns and \(R\) is upper triangular.

Solution Applying the Gram-Schmidt process to the columns \(a_1\) and \(a_2\) of \(A\), we get

\[
v_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\]

\[
v_2 = a_2 - \frac{a_2^T v_1}{||v_1||^2} v_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}
\]

\[
q_1 = \frac{1}{||v_1||} v_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}
\]

\[
q_2 = \frac{1}{||v_2||} v_2 = (1) v_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}
\]

Therefore,

\[
Q = \begin{bmatrix}
1/2 & 1/2 \\
1/2 & -1/2 \\
1/2 & -1/2 \\
1/2 & 1/2
\end{bmatrix}
\]

and \( R = Q^T A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \)
Problem 7 (12 pts).
Are the following assertions true or false? Give a counter-example if the assertion is false.

a. If $A$ is a 4 by 4 matrix then $\det(-3A) = -3^4 \det(A)$.

Solution False. $\det(-3A) = (-3)^4 \det(A) = 3^4 \det(A)$.

b. If $A$ and $B$ are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$.

Solution False. Suppose $A = B$ then $\det(A + B) = \det(2A) = 2^n \det(A)$ which is different from $2 \det(A)$ when $n > 1$.

c. The determinant of a rank one matrix is \textit{not always} zero.

Solution False. If $A$ is a rank one matrix then all the rows are the same. Therefore, $\det(A) = 0$ by Gaussian elimination.