Introduction to the geometry of hamiltonian diffeomorphisms

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1. Preliminaries

1.1 A *symplectic form* on a smooth manifold M is non-degenerate closed 2-form ω. "Non-degenerate" means that the mapping $\tilde{\omega} : T(M) \to T^*(M),\ X \mapsto \tilde{\omega}(X)$ where $\tilde{\omega}(X)(Y) = \omega(X, Y)$ is an isomorphism. We denote the 1-form $\tilde{\omega}(X)$ by $i(X)\omega$.

The couple (M, ω) of a smooth manifold M and a symplectic form ω is called a symplectic manifold. Any symplectic manifold is even dimensional and if $dim(M) = 2n$, ω^n is a volume-form. Hence M is oriented.

Any smooth function $f : M \to \mathbb{R}$ gives rise to a vector field X_f defined uniquely by the equation

$$i(X_f)\omega = df$$

This vector field is called the "Hamiltonian vector field " with hamiltonian f. Let us suppose that f has compact support, (or M is compact). Or more generally, assume that X_f is complete. Then X_f generates a flow ϕ_t^f of M such that $(\phi_t^f)^*\omega = \omega$. Indeed $L_{X_f}\omega = di(X_f)\omega + i(X_f)d\omega = d(df) = 0$. We see that (ϕ_t^f) preserves ω.

A *symplectic diffeomorphism* or *symplectomorphism* of a symplectic manifold (M, ω) is a C^∞ diffeomorphism $h : M \to M$ such that $h^*\omega = \omega$. The support of a diffeomorphism h is the closure of $\{x \in M | h(x) \neq x\}$. The set of all symplectomorphisms with compact support form a group, denoted $Symp(M, \omega)$ (with the law of composition of mappings).

Hence we see that any smooth function with compact support gives rise to a symplectomorphism with compact support.

Let $\mathcal{H}(M, \omega)$ be the group generated by all ϕ_1^f, and their inverse for all smooth functions f with compact support. This an "infinite dimensional " group of symlectomorphisms.

1.2. *Examples of symplectic manifolds*

1. $(\mathbb{R}^{2n}, \omega_0 = \sum_i^n dx_i \wedge dy_i)$

2. The torus T^{2n}. The symplectic form on $\mathbb{R}^{2n}$ is invariant by translations and therefore induces a symplectic form on $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$.

3. The cotangent space $M = T^*(N)$ to any smooth manifold N: The *Liouville 1-form* on $T^*(N)$, is the 1-form λ defined as follows: let $(p, q) \in T^*(N)$ and $\xi \in T_{(p,q)}T^*(N)$, then $\lambda(p, q)(\xi) = p(\pi_*\xi)$, where $\pi : T^*N \to N$ is the canonical projection. Then $\omega = d\lambda$ is a symplectic form.

4. Any oriented surface, with its volume form.

5. The cartesian product $M_1 \times M_2$ of 2 symplectic manifolds (M_i, ω_i) carries the following symplectic form $\omega_1 \ominus \omega_2 = \pi_1^*\omega_1 - \pi_2^*\omega_2$, where π_i are the projections of $M_1 \times M_2$ on each factor.

6. On $(\mathbb{R}^{2n+2}, \omega = \sum_i^{n+2} dx_i \wedge dy_i)$, consider the function $f(x_1, .., x_{n+1}, y_1, .., y_{n+1}) = \sum x_i^2 + y_i^2$. Then $X_f = \sum_i y_i\partial/\partial x_i - x_i\partial/\partial y_i$. Setting $z_k = x_k + iy_k$, we see that the hamilton equations:

$$\begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = -x_i \end{cases}$$

become $\dot{z}_k = -iz_k$. Hence the flow ϕ_t^f is the family of diffeomorphisms $\phi_t^f(z) = z(0)e^{-it}$, $z = (z_1, .., z_n)$.

This flow of X_f induces a action of the circle S^1 on the sphere S^{2n+1}. The quotient of this action is the complex projective space CP^n.

Execice 1 Show that the symplectic form on $\omega = \sum_i dx_i \wedge dy_i$ on $(\mathbb{R}^{2n+2}$ induces a symplectic form on CP^n.

Exercise 2 Show that if α is a 1-form on a smooth manifold N, then $\alpha^*\lambda = \alpha$ where λ is the Liouville 1-form on $T^*(N)$

Exercise 3 Sow that if θ is a closed 2-form on N, then $\omega_\theta = d\lambda + \pi^*\theta$ is a symplectic form on $T^*(N)$. Here $\pi : T^*N \to N$ is the canonical projection.

If θ is exact, show that there exists a diffeomorphism $\phi : T^*(N) \to T^*(N)$ such that $\phi^*(\omega_\theta = d\lambda$.

The first important theorem in symplectic geometry is the following:

Darboux theorem.

Let (M, ω) be a 2n dimensional symplectic manifold. For each point $x \in M$, there is a local chart (U, ϕ) where U is an open neigborhood of x, and $\phi : U \rightarrow \mathbb{R}^{2n}$ is a diffeomorphism such that $\phi^*(\sum_i^n dx_i \wedge dy_i) = \omega_{|U}$.

This theorem says that all symplectic manifolds look alike locally. Therefore there are no symplectic local invariants. All symplectic invariant are of a global nature. We will describe some in these lectures.

The most obvious invariant of a symplectic manifold (M, ω) is the cohomology class $[\omega] \in H^2(M, \mathbb{R})$ of the symplectic form ω. The next important invariant is the group of automorphisms of the symplectic structure. We study this group here and exhibit some invariants emerging from it.

2. Introducing $Sym(M,\omega)_0$ and $Ham(M,\omega)$

We will give $Symp(M,\omega)$ the C^∞-compact-open tologoy. This is the tologoy of uniform convergence over all compact subsets of $h \in Symp(M,\omega)$, and all its partial derivatives (in local charts).

An diffeomorphism ϕ is said to be *isotopic to the identity* if there exists a smooth map $H : M \times [0,1] \to M$ such that if $h_t : M \to M$ is given by $h_t(x) = H(x,t)$, then h_t is a C^∞ diffeomorphism , $h_0 = id_M$ and $h_1 = \phi$. We say that h_t is an isotopy from ϕ to the identity.

A symplectomorphism $\phi \in Symp(M,\omega)$ is isotopic to the identity if in the definition above, h_t is a symplectomorphism for all t. We will say that h_t is a symplectic isotopy (with compact support) form ϕ to the identity. We consider the space $Symp(M,\omega)_0$ of symplectomorphisms which are isotopic to the identity. One shows that this is group, which coincides with the identity component in $Symp(M,\omega)$.

An isotopy h_t of a manifold gives rise to a family of vevtor fields $\dot{h}_t$ defined by

$$\dot{h}_t(x) = \frac{dh_t}{dt}(h_t^{-1}(x))$$

Conversely any family of vector fields X_t with compact support gives rise to an isotopy ϕ_t, via the existence and uniqueness of solutions of ODE theorem:

$$\frac{d\phi_t}{dt}(x) = X_t(\phi_t(x)),\ \phi_0(x) = x$$

If h_t is a symplectic isotopy, then $h_t^*\omega = \omega$. Differentiating this equation gives

$$L_{\dot{h}_t}\omega = 0 \tag{1}$$

where L_X is the Lie derivative in direction X. By Cartan formula $L_X\alpha = i(X)d\alpha + d(i(X)\alpha)$, and the fact that $d\omega = 0$, the equation (1) says that

$$i(\dot{h}_t)\omega$$

is a closed 1-form.

We say that h_t is a *hamiltonian isotopy* if there exists a smooth family of functions $H_t : M \to \mathbb{R}$ such that

$$i(\dot{h}_t)\omega = dH_t \tag{2}$$

A symplectomorphism $\phi \in Symp(M,\omega)$ is said to be a *hamiltonian diffeomorphism* if there exists a hamiltonian isotopy h_t such that $\phi = h_1$. Let $Ham(M,\omega)$ denote the set of all hamiltonian diffeomorphisms of (M,ω).

Exercise 4. Check that $Ham(M,\omega)$ is a normal subgroup of $Symp(M,\omega)$.

Hint Prove first that if h_t, g_t are 2 isotopies, then if $u_t = h_t g_t$

$$\dot{u}_t = \dot{h}_t + (h_t)_* \dot{g}_t \tag{3}$$

This is obtained by the "chain rule" in calculus.

Remark The group $Ham(M,\omega)$ contains the group $\mathcal{H}(M,\omega)$ which is itself very large.

The groups above $Symp(M,\omega)$ and $Ham(M,\omega)$ depend of course on ω. If there exists a diffeomorphism $h : (M,\omega) \to (M',\omega')$ between 2 symplectic manifolds such that $h^*\omega' = \lambda\omega$ for some constant λ, then $I_h : Symp(M,\omega) \to Symp(M',\omega')$ $\phi \mapsto h\phi h^{-1}$ is an isomorphism. The converse is a deep theorem

Theorem 1 (Banyaga).
Let $w : Symp(M,\omega)_0 \to Symp(M',\omega')_0$ or $w : Ham(M,\omega)_0 \to Ham(M',\omega')_0$ group isomorphisms. Then there exists a diffeomorphism $h : M \to M'$ such that $h^*\omega' = \lambda\omega$ for some constant λ and such that $w = I_h$.

This theorem means that the groups $Symp(M,\omega)$ and $Ham(M,\omega)$ determine the symplectic geometry. This theorem is a generalization of a theorem of Filipkiewicz asserting that the group of diffeomorhisms determine the smooth structure.

We have the following important property which is easier to prove:

Theorem 2 (Boothby).
Given two sets $\{x_1,..,x_n\}$ and $\{y_1,..,y_n\}$ of disjoint points on a connected symplectic manifold (M,ω), there is $h \in Ham(M,\omega)$ such that $h(x_i) = y_i$.

This means that any symplectic manifold is a "homogeneous space"

$$M \approx Ham(M,\omega)/Ham(M,\omega)_a$$

where $Ham(M,\omega)_a$ is the isotropy subgroup of some point a.

Two symplectic isotopies h_t, g_t with $g_1 = h_1 = \phi$ are said to be homotopic relatively to ends if there exists a 2 - parameter family of symplectic isotopies $K_{(s,t)}$ such that $K_{(0,t)} = h_t$, $K_{(1,t)} = g_t$ and $K_{(s,1)} = \phi$ for all s. This is an equivalence relation among symplectic isotopies from ϕ to the identity. The set of all equivalence classes $[h_t]$ of symplectic isotopies h_t in $Symp(M,\omega)_0$ is the universal cover $\tilde{S}ymp(M,\omega)_0$ of $Symp(M,\omega)_0$.

For any symplectic isotopy (ϕ_t), we consider the 1-form

$$\Sigma(\phi_t) = \int_0^1 i(\dot{\phi}_t)\omega dt \tag{4}$$

Exercise 5 Show that if two symplectic isotopies h_t, g_t are homotopic relatively to ends, then

$$\Sigma(h_t) - \Sigma(g_t) = d\theta_t$$

for some 1-form θ_t.

This means that we get a well defined map $[h_t] \mapsto [\Sigma(h_t)] \in H^1(M, \mathbb{R})$ from the univeral covering of $Symp(M, \omega)$ into the first de Rham cohomology group of M.

Denote the map above by :

$$\tilde{S} : Sym\tilde{p}(M, \omega \to H^1(M, \mathbb{R})$$

Exercise 5 Show that $\tilde{S}$ is a surjective homomorphism.
Hint : Use exercise 3.

The group

$$\Gamma = \tilde{S}(\pi_1(Symp(M, \omega)_0))) \tag{5}$$

is called the *the flux group.*

We get an induced surjective homomorphism

$$S : Symp(M, \omega)_0 \to H^1(M, \mathbb{R})/\Gamma$$

Remarks

1. The groups $\mathcal{H}(M, \omega)$ and $Ham(M, \omega)$ are contained in the kernel of S.

2. Suppose $h \in Symp(M, \omega)_0$ can be written as $h = h_1....h_N$ where each h_i has compact support in contractible open set, then $S(h) = 0$. Let $\mathcal{H}_0(M, \omega)$ be the group generated by symplectomorphisms with compact supports in contractible open sets. Then $\mathcal{H}_0(M, \omega) \subset KerS$.

We have the following deep results

Theorem 3 (Ono).
The flux group Γ is discret, i.e. $Ham(M, \omega)$ is a closed subgroup of $Symp(M, \omega)_0$ with the C^∞ topology.

Theorem 4 (Banyaga).

Let (M, ω) be a connected symplectic manifold,

1) $Ham(M, \omega) = KerS$

2) If M is compact, $Ham(M, \omega) = KerS = [Symp(M, \omega)_0 Symp(M, \omega)_0]$ is a simple group

Here we denoted by $[G, G]$ the commutator subgroup of a group G.

Corollary

Let (M, ω) be a compact symplectic manifold. Then

$$\mathcal{H}(M, \omega) = \mathcal{H}_0(M, \omega) = Ham(M, \omega) = KerS$$

This theorem implies that there is no non-trivial homomorphism from $Ham(M, \omega)$ into any abelian group.

3. The local structure

A submanifold L of a symplectic manifold (M, ω) is called a *lagrangian* submanifold if $i^*\omega = 0$ where $i : L \to M$ is the inclusion.

Exercise 6 Show that the graph of a 1-form α on N into T^*N is a lagrangian submanifold iff α is closed.

Hint : Use exercise 2.

Exercise 7 Let $h : M \to M$ be a symplectomorphism of (M, ω) show that its graph in $M \times M, \omega \ominus \omega)$ is a lagrangian submanifold.

Exercise 8. Let $\mathcal{L}_\omega(M)$ be the set of symplectic vector fields, i.e. vector fields X such that $i(X)$ is a closed form. Show that $\mathcal{L}_\omega(M)$ is a Lie algebra. Show that the map : $s : \mathcal{L}_\omega(M) \to H^1(M, \mathbb{R})$, $X \mapsto [i(X)\omega]$ is a surjective Lie algebra homomorphism and its kernel is the derived Lie algebra $[\mathcal{L}_\omega(M)\mathcal{L}_\omega(M)]$, (generated by commutators).

Local structure The diagonal $\Delta \subset (M \times M, \omega \ominus \omega)$ is a lagrangian submanifold (the graph of the identity). A theorem of Kostant-Sternberg-Weinstein says that there exists a diffeomorphism k from a neigborhood $\mathcal{U}(\Delta)$ of Δ in $M \times M$ onto $T^*(\Delta) \approx T^*(M)$ such that $k_{|\Delta \approx M}$ is the identity and $k^*\omega_M = \omega \ominus \omega$.

If h is a symplectomorphism C^1 close enough to the identity, and its graph $\Gamma(h)$ fits inside the neigborhood $\mathcal{U}()$, then $k(\Gamma(h)$ is a lagrangian submanifold of $T^*(M)$, which is C^1 close to the diagonal ;it is then the grapg of a 1-form $W(h)$. This form is closed. The correspondance $h \mapsto W(h)$ is a smooth chart of a neigborhood of the identity in $Symp(M, \omega)_0$, called the Weinstein chart.

The existence of this chart implies that $Symp(M,\omega)_0$ is smoothly locally contractible, and locally connected by smooth arcs. It is clear that the Weinstein chart takes a small neigborhood of the identity in $Ham(M,\omega)$ to the space $B^1(M)$ of exact 1-forms.

Theorem 5 (Weinstein).
A C^1 small hamiltonian diffeomorphism h on compact manifold has as many fixed points as can have a smooth function of a compact manifold.

Proof.
Indeed the zeros of the Weinstein form $W(h)$ correspond to the intersection of the graph of h with the diagonal, i.e correspond to fixed points of h. But since $W(h) = df$, these zero are the critical points of f. □

Arnold Conjecture

Theorem 4 is a particular case of a general conjecture made by Arnold in the 60's. It says that *if h is a hamiltonian diffeomorphism of a compact symplectic manifold (M,ω) such that its graph $\Gamma(h)$ intersects the diagonal transversally, then the number of its fixed points is no smaller than the number of critical points a smooth function is allowed to have.*

Another formulation is that the number of fixed points is bounded from below by the sum of the Betti numbers of M.

This conjecture is solved nowaday. It has been a driving force which led to tremendous developpement in "Symplectic Topology".

One first observe that the set of fixed points of a hamiltonian diffeomorphisms are in 1-1 correspondance with critical points of a functional " the ıaction-functional " on the infinite dimensional space " of contractible loops on M. *Floer homology* is the homology whoee chains are generated by these critical points. The main result is that *Floer Homology* is isomorphic to the singular homology. The Arnold conjecture then follows.

4. Entering Symplectic Topology

4.1 Symplectic capacities

Let (M, ω) be a 2n dimensional symplectic manifold. It is obvious that any symplectomorphism h preserve the Liouville volume $\Omega = \omega^n$. *Symplectic Topology* emerged from the search of invariants which distinguish "volume preserving properties" from symplectic properties of symplectic manifolds of dimension $2n \geq 4$.

Darboux' theorem says that near every point in (M, ω), there is a local diffeomorphism of a small ball of radius r, centered at the origin $B(0, r)$ in $\mathbb{R}^{2n}$ such that $\phi^*(\omega_0) = \omega_{|}\phi^{-1}(B(0, r)$ where $\omega_0 = sumdx_i \wedge dy_i$.

Let us look at the largest ball that can be embedded in M., i.e. we define:

$$D(M, \omega) = sup\{\pi r^2\}$$

where r runs over all the radii of balls $B(0, r)$ that can be embedded symplectically in (M, ω).

This is a symplectic invariant called the *Gromov width.* , or the *Gromov capacity.*

Definition Let $c : \mathcal{S}(2n)$ be the class of all 2n dimensional symplectic manifolds.
A *capacity* is a map $c : \mathcal{S}(2n) \to \mathbb{R} \cup \infty$ such that
(1) $c(M, \omega) \leq c(N, \tau)$ if there exists a symplectic emdebbing $\phi : (M, \omega) \to (N, \tau)$. (monotonicity)

(2) $c(M, \alpha\omega) = |\alpha| c(M, \omega)$ for all non-zero number α.
(conformality)

(3) $c(B(0, 1), \omega_0) = c(Z(1), \omega_0) = \pi$
where
$Z(r) = \{x_1, .., x_n, y_1, ...y_n)|x_1^2 + y_1^2 \leq r^2]$

If n= 1, then $c(M, \omega) = |\int_M \omega|$ is a capacity. If $n \geq 1$, the condition (3) excludes this example.

Theorem 6 (Gromov).
The Gromov width is a symplectic capacity.

Gromov used this fact to get the famous

Non sqeezing theorem (Gromov).
If there is a symplectic embedding from $Z(r) \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ into the ball $B(0, R)$, then $r \leq R$.

This theorem has been populalized as " the Gromov camel".

Proof.
Immediate from the existence of symplectic capacities. □

4.2 The Hofer metric

For $\phi \in Ham(M,\omega)$, choose a hamiltonian isotopy $\Phi = (\phi_t)$ from ϕ to the identity.

Hofer defined the length of this isotopy

$$l_H(\Phi) = \int_0^1 osc(F_t)dt \tag{8}$$

where $i(\dot{\phi_t}) = dF_t$ and $osc(H) = maxH(x) - minH(x)$.

Exercise 10 The length functin satisfies :
(i) $l_H(\Phi) \geq 0$
(ii) $l_H(\phi.\phi') \leq l_H(\Phi) + l_H(\Phi')$
(iii) $l_H(\Phi) = l_H(\Phi^{-1})$
(iv) $l_H(h.\Phi.h^{-1}) = l_H(\Phi)$ for all sympectomorphism h.

Here $\Phi.\Phi' = (\phi_t\phi'_t)$.

Consider

$$\nu(\phi) = inf(l_H(\Phi)) \tag{9}$$

where the infimum is taken over all hamiltoinan isotopies from ϕ to the identity.

Theorem 8 (Hofer, Viterbo, Polterowich, Lalonde-McDuff).
The function $e(\phi)$ is a bi-invariant metric on $Ham(M,\omega)$, i.e. The following are satisfied
(i) $\nu(\phi) \geq 0$ and
(i') $\nu(\phi) = 0$ implies that $\phi = id_M$
(ii) $\nu(\phi) = \nu(\phi^{-1})$
(iii) $\nu(\phi) \leq \nu(\phi) + \nu(\psi)$
(iv) $\nu(h\phi.h^{-1}) = \nu(\phi)$

Therefore the function on $Ham(M,\omega)$

$$d(\phi,\psi) = \nu(\phi.\psi^{-1}) \tag{10}$$

is a bi-invariant distance on $Ham(M,\omega)$. We call $\nu(\phi) = ||\phi||$ the Hofer norm of ϕ and $d(\phi,\psi)$ the Hofer distance from ϕ to ψ.

Hofer geometry is the geometry of $Ham(M,\omega)$ equipped with the Hofer metric, and *Hofer topology* is the topology induced by the Hofer distance. This geometry and topology are not well understood.

In theorem 8, propery (i') is very difficult to prove. We give below the main arguments. The other properties come straight form exercise 10.

Definition The *displacement energy* of a bounded subset A of M is given by:

$$e(A) = inf\{||\phi||, \phi \in Ham(M, \omega), \phi(A) \cap A = \emptyset\} \tag{11}$$

Theorem 9 (Eliashberg-Polterovich.
For any non-empty open set A, $e(A)$ is strictly positive.

The connection between the energy, the capacity is given by this

Theroem 10 : Energy-capacity inequalitY ((Hofer, Lalonde-McDuff).
$sup\{c(U)|U \text{ open and } \phi(U) \cap U = \emptyset\} \leq \nu(\phi)$

Ecxercise 11 Use theorem 10 to prove property (i') in theorem 8.

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