SPRING 2009

MATH 401 - NOTES

Sequences of functions
Pointwise and Uniform Convergence

Previously, we have studied sequences of *real numbers*. Now we discuss *sequences of real-valued functions*. By a sequence $\{f_n\}$ of real-valued functions on D, we mean a sequence $(f_1, f_2, \dots, f_n, \dots)$ such that each f_n is a function having domain D and range a subset of $\mathbb{R}$.

I. Pointwise convergence

Definition. *Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of functions defined on D. We say that $\{f_n\}$ converges pointwise on D if*

$$\lim_{n\to\infty} f_n(x) \text{ exists for each point } x \text{ in } D.$$

In other words, $\lim_{n\to\infty} f_n(x)$ must be a real number that depends only on x.

In this case, we write

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for every x in D and f is called the *pointwise limit of the sequence* $\{f_n\}$.

Formal Definition: The sequence $\{f_n\}$ converges pointwise to f on D if for every $x \in D$ and for every $\epsilon > 0$, there exists a natural number $N = N(x, \epsilon)$ such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{whenever} \quad n > N.$$

Note: The notation $N = N(x, \varepsilon)$ means that the natural number N depends on the choice of x and ε.

Example 1. Let $\{f_n\}$ be the sequence of functions on $\mathbb{R}$ defined by $f_n(x) = nx$. This sequence does not converge pointwise on $\mathbb{R}$ because $\lim_{n\to\infty} f_n(x) = \infty$ for any $x > 0$.

Example 2. Let $\{f_n\}$ be the sequence of functions on $\mathbb{R}$ defined by

$$f_n(x) = \frac{x}{n}.$$

This sequence converges pointwise to the zero function on $\mathbb{R}$. Indeed, given any $\epsilon > 0$, choose $N > \frac{x}{\epsilon}$ then

$$|f_n(x) - 0| = \frac{x}{n} < \frac{x}{N} < \epsilon, \quad \text{for } n > N$$

Example 3. *Consider the sequence $\{f_n\}$ of functions defined by*

$$f_n(x) = \frac{(x+n)^2}{n^2} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Show that $\{f_n\}$ converges pointwise.

Solution: For every real number x, we have:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\frac{x^2}{n^2} + \frac{2x}{n} + 1 \right) = x^2 \left(\lim_{n\to\infty} \frac{1}{n^2} \right) + 2x \left(\lim_{n\to\infty} \frac{1}{n} \right) + 1 = 0 + 0 + 1 = 1$$

Thus, $\{f_n\}$ converges pointwise to the function $f(x) = 1$ on $\mathbb{R}$.

Example 4. *Consider the sequence $\{f_n\}$ of functions defined by $f_n(x) = n^2 x^n$ for $0 \leq x \leq 1$. Determine whether $\{f_n\}$ is pointwise convergent on $[0, 1]$.*

Solution: First of all, we observe that $f_n(0) = 0$ for every n in $\mathbb{N}$. So the sequence $\{f_n(0)\}$ is constant and converges to zero. Now suppose $0 < x < 1$ then $n^2 x^n = n^2 e^{n \ln(x)} \to 0$ as $n \to \infty$. Finally, $f_n(1) = n^2$ for all n. So, $\lim_{n\to\infty} f_n(1) = \infty$. Therefore, $\{f_n\}$ is not pointwise convergent on $[0, 1]$. Although, it is pointwise convergent on $[0, 1)$.

Example 5. *Consider the sequence $\{f_n\}$ of functions defined by*

$$f_n(x) = \frac{\sin(nx + 3)}{\sqrt{n+1}} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Show that $\{f_n\}$ converges pointwise.

Solution: For every x in $\mathbb{R}$, we have

$$\frac{-1}{\sqrt{n+1}} \leq \frac{\sin(nx+3)}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}}$$

Moreover,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0.$$

Applying the sandwich theorem for sequences, we obtain that

$$\lim_{n\to\infty} f_n(x) = 0 \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Therefore, $\{f_n\}$ converges pointwise to the function $f = 0$ on $\mathbb{R}$.

Example 6. *Let $\{f_n\}$ be the sequence of functions defined by $f_n(x) = \cos^n(x)$ for $-\pi/2 \leq x \leq \pi/2$. Discuss the pointwise convergence of the sequence.*

Solution: For $-\pi/2 \leq x < 0$ and for $0 < x \leq \pi/2$, we have

$$0 \leq \cos(x) < 1.$$

It follows that

$$\lim_{n\to\infty} (\cos(x))^n = 0 \qquad \text{for } x \neq 0.$$

Moreover, since $f_n(0) = 1$ for all n in $\mathbb{N}$, one gets $\lim_{n\to\infty} f_n(0) = 1$. Therefore, $\{f_n\}$ converges pointwise to the function f defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad -\frac{\pi}{2} \leq x < 0 \quad \text{or} \quad 0 < x \leq \frac{\pi}{2} \\ 1 & \text{if} \qquad x = 0 \end{cases}$$

Example 7. *Consider the sequence $\{f_n\}$ of functions defined by*

$$f_n(x) = \frac{x}{3 + nx^2} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Show that $\{f_n\}$ converges pointwise.

Solution: Moreover, for every real number x, we have:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{3 + nx^2} = 0.$$

Hence, $\{f_n\}$ converges pointwise to the zero function.

Example 8. *Consider the sequence of functions defined by*

$$f_n(x) = nx(1 - x)^n \quad \text{on } [0, 1].$$

Show that $\{f_n\}$ converges pointwise to the zero function.

Solution: Note that $f_n(0) = f_n(1) = 0$, for all $n \in \mathbb{N}$. Now suppose $0 < x < 1$, then

$$\lim_{n\to\infty} f_n(x) = 0$$

Therefore, the given sequence converges pointwise to zero.

Example 9. *Let $\{f_n\}$ be the sequence of functions on $\mathbb{R}$ defined by*

$$f_n(x) = \begin{cases} n^3 & \text{if} \quad 0 < x \le \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Show that $\{f_n\}$ converges pointwise to the constant function $f = 1$ on $\mathbb{R}$.

Solution: For any x in $\mathbb{R}$ there is a natural number N such that x does not belong to the interval $(0,\ 1/N)$. The intervals $(0,\ 1/n)$ get smaller as $n \to \infty$. We see that $f_n(x) = 1$ for all $n > N$. Hence,

$$\lim_{n\to\infty} f_n(x) = 1 \quad \text{for all } x.$$

II. Uniform convergence

Definition. Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of real valued functions defined on D. Then $\{f_n\}$ converges uniformly to f if given any $\epsilon > 0$, there exists a natural number $N = N(\epsilon)$ such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for every } n > N \quad \text{and for every } x \text{ in } D.$$

Note: In the above definition the natural number N depends only on ε. Therefore, uniform convergence implies pointwise convergence. But the converse is false as we can see from the following counter-example.

Example 10 *Let $\{f_n\}$ be the sequence of functions on $(0,\ \infty)$ defined by*

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

This sequence converges pointwise to zero. Indeed, $(1 + n^2x^2) \sim n^2x^2$ as n gets larger and larger. So,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{n^2x^2} = \frac{1}{x} \lim_{n\to\infty} \frac{1}{n} = 0.$$

But for any $\varepsilon < 1/2$, we have

$$\left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| = \frac{1}{2} - 0 > \varepsilon.$$

Hence $\{f_n\}$ is not uniformly convergent.

Theorem. *Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of* **continuous** *functions on D which converges uniformly to f on D. Then its limit f is continuous on D.*

Example 10. *Let $\{f_n\}$ be the sequence of functions defined by $f_n(x) = \cos^n(x)$ for $-\pi/2 \leq x \leq \pi/2$. Discuss the uniform convergence of the sequence.*

Solution: We know that $\{f_n\}$ converges pointwise to the function f defined by (see Example 6)

$$f(x) = \begin{cases} 0 & \text{if} \quad -\frac{\pi}{2} \leq x < 0 \quad \text{or} \quad 0 < x \leq \frac{\pi}{2} \\ 1 & \text{if} \quad x = 0 \end{cases}$$

Each $f_n(x) = \cos^n(x)$ is continuous on $[-\pi/2,\ \pi/2]$. But the pointwise limit is not continuous at $x = 0$. By the above theorem, we conclude that $\{f_n\}$ does not converge uniformly on $[-\pi/2,\ \pi/2]$.

Example 11. *Consider the sequence $\{f_n\}$ of functions defined by*

$$f_n(x) = \frac{\sin(nx+3)}{\sqrt{n+1}} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Prove that $\{f_n\}$ converges uniformly to the zero function on $\mathbb{R}^n$.

Solution: We have seen that $\{f_n\}$ converges pointwise to the zero function on $\mathbb{R}^n$ (see Example 5). Moreover

$$|f_n(x) - 0| = \frac{|\sin(nx+3)|}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}}.$$

Given any $\epsilon >$, we can find $N \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{n+1}} < \epsilon \quad \text{whenever} \quad n > N.$$

It follows $|f_n(x) - f(x)| < \epsilon$ for every $n > N$ and for every x in$\mathbb{R}$. Hence, $\{f_n\}$ converges uniformly to the zero function on $\mathbb{R}$.