Reduction Theory

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OVERVIEW OF THE COURSE

• Lie group actions

• Symplectic manifolds

• Poisson manifolds

• Abstract symmetry reduction
• Conservation laws via generalized distributions

• The optimal momentum map and groupoids

• Optimal reduction

• Singular point reduction
• Singular orbit reduction

• Poisson reduction

• Coisotropic reduction

• Cosymplectic reduction

• Dirac structures
$M$ a manifold and $G$ a Lie group. A **left action** of $G$ on $M$ is a smooth mapping $\Phi : G \times M \to M$ such that

(i) $\Phi(e, z) = z$, for all $z \in M$ and

(ii) $\Phi(g, \Phi(h, z)) = \Phi(gh, z)$ for all $g, h \in G$ and $z \in M$.

We will often write

$$g \cdot z := \Phi(g, z) := \Phi_g(z) := \Phi^z(g).$$
and

\[ A_G := \{ \Phi_g \mid g \in G \} \subset \text{Diff}(M). \]

The triple \((M, G, \Phi)\) is called a \textit{G-space} or a \textit{G-manifold}.

**Examples of group actions**

- **Translation and conjugation.** The left (right) translation \( L_g : G \to G, \ (R_g) \ h \mapsto gh \), induces a left (right) action of \(G\) on itself.

- **The inner automorphism** \( \text{AD}_g : G \to G \), given by \( \text{AD}_g := R_{g^{-1}} \circ L_g \) defines a left action of \(G\) on itself called \textit{conjugation}. 
- **Adjoint and coadjoint action.** The differential at the identity of the conjugation mapping defines a linear left action of $G$ on $\mathfrak{g}$ called the **adjoint representation** of $G$ on $\mathfrak{g}$

$$\text{Ad}_g := T_e \text{AD}_g : \mathfrak{g} \to \mathfrak{g}.$$ 

If $\text{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual of $\text{Ad}_g$, then the map

$$\Phi : G \times \mathfrak{g}^* \to \mathfrak{g}^* \quad (g, \nu) \mapsto \text{Ad}^*_{g^{-1}} \nu,$$

defines also a linear left action of $G$ on $\mathfrak{g}^*$ called the **coadjoint representation** of $G$ on $\mathfrak{g}^*$. 
• **Group representation.** If the manifold $M$ is a vector space $V$ and $G$ acts linearly on $V$, that is, $\Phi_g \in \text{GL}(V)$ for all $g \in G$, where $\text{GL}(V)$ denotes the group of all linear automorphisms of $V$, then the action is said to be a **representation** of $G$ on $V$. For example, the adjoint and coadjoint actions of $G$ defined above are representations.

• **Tangent lift of a group action.** $\Phi$ induces a natural action on the tangent bundle $TM$ of $M$ by

$$g \cdot v_m := T_m\Phi_g(v_m), \quad g \in G, \quad v_m \in T_mM.$$
Cotangent lift of a group action. Let $\Phi : G \times M \to M$ be a smooth Lie group action on the manifold $M$. The map $\Phi$ induces a natural action on the cotangent bundle $T^*M$ of $M$ by

$$g \cdot \alpha_m := T_{g \cdot m} \Phi_{g^{-1}}(\alpha_m)$$

where $g \in G$ and $\alpha_m \in T^*_m M$. 
The **infinitesimal generator** $\xi_M \in \mathfrak{X}(M)$ associated to $\xi \in \mathfrak{g}$ is the vector field on $M$ defined by

$$\xi_M(m) := \frac{d}{dt} \bigg|_{t=0} \Phi_{\exp t}\xi(m) = T_{e\Phi^m} \cdot \xi.$$ 

The infinitesimal generators are complete vector fields. The flow of $\xi_M$ equals $(t, m) \mapsto \exp t\xi \cdot m$. Moreover, the map $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ is a **Lie algebra antihomomorphism**, that is,

1. (i) $(a\xi + b\eta)_M = a\xi_M + b\eta_M$,

2. (ii) $[\xi, \eta]_M = -[\xi_M, \eta_M]$. 
Let $\mathfrak{g}$ be a Lie algebra and $M$ a smooth manifold. A right (left) Lie algebra action of $\mathfrak{g}$ on $M$ is a Lie algebra (anti)homomorphism $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ such that the mapping $(m, \xi) \in M \times \mathfrak{g} \mapsto \xi_M(m) \in TM$ is smooth.

Given a Lie group action, we will refer to the Lie algebra action induced by its infinitesimal generators as the associated Lie algebra action.
Stabilizers and orbits. The isotropy subgroup or stabilizer of an element $m$ in the manifold $M$ acted upon by the Lie group $G$ is the closed subgroup

$$G_m := \{ g \in G \mid \Phi_g(m) = m \} \subset G$$

whose Lie algebra $\mathfrak{g}_m$ equals

$$\mathfrak{g}_m = \{ \xi \in \mathfrak{g} \mid \xi_M(m) = 0 \}.$$ 

The orbit $O_m$ of the element $m \in M$ under the group action $\Phi$ is the set

$$O_m \equiv G \cdot m := \{ \Phi_g(m) \mid g \in G \}.$$
The isotropy subgroups of the elements in a group orbit are related by the expression

\[ G_{g \cdot m} = gG_m g^{-1} \text{ for all } g \in G. \]

The notion of orbit allows the introduction of an equivalence relation in the manifold \( M \), namely, two elements \( x, y \in M \) are equivalent if and only if they are in the same \( G \)–orbit, that is, if there exists an element \( g \in G \) such that \( \Phi_g(x) = y \). The space of classes with respect to this equivalence relation is usually referred to as the space of orbits and, depending on the context, it is denoted by the symbols \( M/G \) or \( M/A_G \).
- **Transitive action**: only one orbit, that is, $\mathcal{O}_m = M$

- **Free action**: $G_m = \{e\}$ for all $m \in M$

- **Proper action**: if $\Phi : G \times M \to M \times M$ defined by

  $$\Phi(g, z) := (z, \Phi(g, z))$$

is proper. This is equivalent to: for any two convergent sequences $\{m_n\}$ and $\{g_n \cdot m_n\}$ in $M$, there exists a convergent subsequence $\{g_{n_k}\}$ in $G$.

Examples of proper actions: compact group actions, $SE(n)$ acting on $\mathbb{R}^n$, Lie groups acting on themselves by translation.
Fundamental facts about proper Lie group actions

Φ : \( G \times M \to M \) be a proper action of the Lie group \( G \) on the manifold \( M \). Then:

(i) The isotropy subgroups \( G_m \) are compact.

(ii) The orbit space \( M/G \) is a Hausdorff topological space (even when \( G \) is not Hausdorff).

(iii) If the action is free, \( M/G \) is a smooth manifold, and the canonical projection \( \pi : M \to M/G \) defines on \( M \) the structure of a smooth left principal \( G \)–bundle.
(iv) If all the isotropy subgroups of the elements of $M$ under the $G$–action are conjugate to a given one $H$ then $M/G$ is a smooth manifold and $\pi : M \to M/G$ defines the structure of a smooth locally trivial fiber bundle with structure group $N(H)/H$ and fiber $G/H$.

(v) If the manifold $M$ is paracompact then there exists a $G$-invariant Riemannian metric on it.

(vi) If the manifold $M$ is paracompact then smooth $G$-invariant functions separate the $G$-orbits.
Twisted product. Let $G$ be a Lie group and $H \subset G$ a subgroup. Suppose that $H$ acts on the left on the manifold $A$. The right twisted action of $H$ on the product $G \times A$ is defined by

$$(g, a) \cdot h = (gh, h^{-1} \cdot a).$$

This action is free and proper by the freeness and properness of the action on the $G$–factor. The twisted product $G \times_H A$ is defined as the orbit space $(G \times A)/H$ corresponding to the twisted action.
Tube. Let $M$ be a manifold and $G$ a Lie group acting properly on $M$. Let $m \in M$ and denote $H := G_m$. A tube around the orbit $G \cdot m$ is a $G$-equivariant diffeomorphism

$$\varphi : G \times_H A \longrightarrow U,$$

where $U$ is a $G$-invariant neighborhood of $G \cdot m$ and $A$ is some manifold on which $H$ acts.
Slice Theorem.  $G$ a Lie group acting properly on $M$ at the point $m \in M$, $H := G_m$. There exists a tube

$$\varphi : G \times_H B \to U$$

about $G \cdot m$. $B$ is an open $H$-invariant neighborhood of 0 in a vector space which is $H$-equivariantly isomorphic to $T_mM/T_m(G \cdot m)$, where the $H$-representation is given by

$$h \cdot (v + T_m(G \cdot m)) := T_m\Phi_h \cdot v + T_m(G \cdot m).$$

Slice: $S := \varphi([e, B])$ so that $U = G \cdot S$. 
Dynamical consequences. \( X \in \mathfrak{X}(U)^G, \ U \subset M \) open \( G \)-invariant, \( S \) slice at \( m \in U \). Then there exists

\begin{itemize}
  \item \( X_T \in \mathfrak{X}(G \cdot S)^G, \ X_T(z) = \xi(z)_M(z) \) for \( z \in G \cdot S \), where \( \xi : G \cdot S \to g \) is smooth \( G \)-equivariant and \( \xi(z) \in \text{Lie}(N(G_z)) \) for all \( z \in G \cdot S \). The flow \( T_t \) of \( X_T \) is given by \( T_t(z) = \exp t\xi(z) \cdot z \), so \( X_T \) is complete.

  \item \( X_N \in \mathfrak{X}(S)^G_m \)

  \item If \( z = g \cdot s \), for \( g \in G \) and \( s \in S \), then

\[ X(z) = X_T(z) + T_s \Phi_g (X_N(s)) = T_s \Phi_g (X_T(s) + X_N(s)) \]
\end{itemize}
• If $N_t$ is the flow of $X_N$ (on $S$) then the integral curve of $X \in \mathfrak{X}(U)^G$ through $g \cdot s \in G \cdot S$ is

$$F_t(g \cdot s) = g(t) \cdot N_t(s),$$

where $g(t) \in G$ is the solution of

$$\dot{g}(t) = T_{eL g(t)}(\xi(N_t(s))), \quad g(0) = g.$$

This is the tangential-normal decomposition of a $G$-invariant vector field (or Krupa decomposition in bifurcation theory).
Geometric consequences. Orbit type, fixed point, and isotropy type spaces

\[ M_{(H)} = \{ z \in M \mid G_z \in (H) \}, \]
\[ M^H = \{ z \in M \mid H \subset G_z \}, \]
\[ M_H = \{ z \in M \mid H = G_z \} \]

are submanifolds.

\[ M_H \] is open in \( M^H \).

\( m \in M \) is regular if \( \exists U \ni m \) such that \( \dim \mathcal{O}_z = \dim \mathcal{O}_m, \forall z \in U \).
**Principal Orbit Theorem:** $M$ connected. The subset $M^{reg}$ is connected, open, and dense in $M$. $M/G$ contains only one principal orbit type, which is a connected open and dense subset of it.

**The Stratification Theorem:** Let $M$ be a smooth manifold and $G$ a Lie group acting properly on it. The connected components of the orbit type manifolds $M^{(H)}$ and their projections onto orbit space $M^{(H)}/G$ constitute a Whitney stratification of $M$ and $M/G$, respectively. This stratification of $M/G$ is minimal among all Whitney stratifications of $M/G$. 
$G$-Codistribution Theorem: Let $G$ be a Lie group acting properly on the smooth manifold $M$ and $m \in M$ a point with isotropy subgroup $H := G_m$. Then

$((T_m(G \cdot m))^\circ)^H = \{ df(m) \mid f \in C^\infty(M)^G \}$. 
SIMPLE EXAMPLES

• $S^1$ acting on $\mathbb{R}^2$

Since $S^1$ is Abelian we do not distinguish between orbit types and isotropy types, that is, $\mathbb{R}^2_{(H)} = \mathbb{R}^2_H$ for any isotropy group $H$ of this action.

If $x \neq 0$ then $S^1_x = 1$ and $S^1 \cdot x$ is the circle centered at the origin of radius $\|x\|$. The slice is the ray through 0 and $x$. $(\mathbb{R}^2)^{\text{reg}} = \mathbb{R}^2 \setminus \{0\}$, which is open, connected, dense. $\mathbb{R}_1^2 = (\mathbb{R}^2)^{\text{reg}}$ and $(\mathbb{R}^2)^{\text{reg}}/S^1 = ]0, \infty[.$
If \( x = 0 \), then \( S^1_0 = S^1 \). The slice is \( \mathbb{R}^2 \). \( \mathbb{R}^2_0 = \{0\} \) and \( \mathbb{R}^2_0/S^1 = \{0\} \).

Finally \( \mathbb{R}^2/S^1 = [0, \infty[. \)

- **SO(3) acting on \( \mathbb{R}^3 \)**

Since SO(3) is non-Abelian, there is a distinction between orbit and isotropy types.

Since every rotation has an axis, if \( x \neq 0 \) the isotropy subgroup \( SO(3)_x = S^1(x) \), the circle representing the rotations with axis \( x \). So \( (\mathbb{R}^3)^{reg} = \mathbb{R}^3 \setminus \{0\} \).
The orbit $\text{SO}(3) \cdot x$ is the sphere centered at the origin with radius $\|x\|$. The slice at $x$ is the ray connecting the origin to $x$.

$(\mathbb{R}^3)_{S^1(x)}$ is the set of points in $\mathbb{R}^3$ which have the same isometry group $S^1(x)$, so it is equal to the line through the origin and $x$ with the origin eliminated. It is disconnected and not $\text{SO}(3)$-invariant.

$(\mathbb{R}^3)_{(S^1(x))}$ is the set of points in $\mathbb{R}^3$ which have the isometry group $S^1(x)$ conjugate to $S^1(x)$. But any two rotations are conjugate, so $(\mathbb{R}^3)_{(S^1(x))} = \mathbb{R}^3 \setminus \{0\}$, which
is again equal in this case to \((\mathbb{R}^3)^{reg}\). This is connected, open, dense. \((\mathbb{R}^3)_{(S^1(x))}/SO(3) = \]0, \infty[.\)

If \(x = 0\), the slice is \(\mathbb{R}^3\), \(SO(3)_0 = SO(3)\), \((\mathbb{R}^3)_{SO(3)} = (\mathbb{R}^3)_{SO(3)} = \{0\}\), and \((\mathbb{R}^3)_{(SO(3))} = \{0\}/SO(3) = \{0\}\).

Finally \(\mathbb{R}^3/\text{SO}(3) = [0, \infty[.\)

- **Semidirect products**

\(V\) vector space, \(G\) Lie group

\(\sigma: G \to \text{GL}(V)\) representation
$\sigma': g \rightarrow \text{gl}(V)$ induced Lie algebra representation:

$$\xi \cdot v := \xi_V(v) := \sigma'(\xi)v := \frac{d}{dt}\Big|_{t=0} \sigma(\exp t\xi)v$$

$S := G\ltimes V$ semidirect product: underlying manifold is $G \times V$, multiplication

$$(g_1, v_1)(g_2, v_2) := (g_1g_2, v_1 + \sigma(g_1)v_2)$$

for $g_1, g_2 \in G$ and $v_1, v_2 \in V$, identity element is $(e, 0)$ and $(g, v)^{-1} = (g^{-1}, -\sigma(g^{-1})v)$.

Note that $V$ is a normal subgroup of $S$ and that $S/V = G$. 
Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\mathfrak{s} := \mathfrak{g} \circledS V$ be the Lie algebra of $S$; it is the semidirect product of $\mathfrak{g}$ with $V$ using the representation $\sigma'$ and its underlying vector space is $\mathfrak{g} \times V$. The Lie bracket on $\mathfrak{s}$ is given by

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \sigma'(\xi_1)v_2 - \sigma'(\xi_2)v_1)$$

for $\xi_1, \xi_2 \in \mathfrak{g}$ and $v_1, v_2 \in V$.

Identify $\mathfrak{s}^*$ with $\mathfrak{g}^* \times V^*$ by using the duality pairing on each factor.
Adjoint action of $S$ on $\mathfrak{s}$:

$$\text{Ad}_{(g,u)}(\xi, v) = (\text{Ad}_g \xi, \sigma(g)v - \sigma'(\text{Ad}_g \xi)u),$$
for $(g, u) \in S, (\xi, v) \in \mathfrak{s}$.

Coadjoint action of $S$ on $\mathfrak{s}^*$:

$$\text{Ad}^*_{(g,u)^{-1}}(\nu, a) = \left(\text{Ad}^*_g^{-1} \nu + (\sigma'_u)^* \sigma_*(g)a, \sigma_*(g)a\right),$$
for $(g, u) \in S, (\nu, a) \in \mathfrak{s}^*$, where

$$\sigma_*(g) := \sigma(g^{-1})^* \in \text{GL}(V^*),$$

$\sigma'_u : \mathfrak{g} \to V$ is the linear map given by $\sigma'_u(\xi) := \sigma'(\xi)u$ and $(\sigma'_u)^* : V^* \to \mathfrak{g}^*$ is its dual.
Classification of orbits is a major problem!

Do the example of the coadjoint action of $SE(3) = SO(3) \circledast \mathbb{R}^3$. In this case:

$\sigma : SO(3) \to GL(\mathbb{R}^3)$ is usual matrix multiplication on vectors, that is, $\sigma(A)v := Av$, for any $A \in SO(3)$ and $v \in \mathbb{R}^3$.

Dualizing we get $\sigma(A)^*\Gamma = A^*\Gamma = A^{-1}\Gamma$, for any $\Gamma \in V^* \cong \mathbb{R}^3$. 
The induced Lie algebra representation $\sigma' : \mathbb{R}^3 \cong \mathfrak{so}(3) \to \mathfrak{gl}(\mathbb{R}^3)$ is given by $\sigma'(\Omega)v = \sigma'_v \Omega = \Omega \times v$, for any $\Omega, v \in \mathbb{R}^3$.

Therefore, $(\sigma'_v)^* \Gamma = v \times \Gamma$ and $\sigma'(\Omega)^* \Gamma = \Gamma \times \Omega$, for any $v \in V \cong \mathbb{R}^3$, $\Omega \in \mathbb{R}^3 \cong \mathfrak{so}(3)$, and $\Gamma \in V^* \cong \mathbb{R}^3$.

We have $\text{ad}_{\Omega}^* \Pi = \Pi \times \Omega$

So all formulas in this case become:

$$(A, a)(B, b) = (AB, Ab + a)$$

$$(A, a)^{-1} = (A^{-1}, -A^{-1}a)$$
\[(x, y), (x', y')\] = (x \times x', x \times y' - x' \times y)\]

\[
\text{Ad}_{(A, a)}(x, y) = (Ax, Ay - Ax \times a)
\]

\[
\text{Ad}^*_{(A, a)}^{-1}(u, v) = (Au + a \times Av, Av)
\]

Let \(\{e_1, e_2, e_3, f_1, f_2, f_3\}\) be an orthonormal basis of \(\mathfrak{se}(3) = \mathbb{R}^3 \times \mathbb{R}^3\) such that \(e_i = f_i\) for \(i = 1, 2, 3\). The dual basis of \(\mathfrak{se}(3)^*\) using the dot product is again \(\{e_1, e_2, e_3, f_1, f_2, f_3\}\).

Let \(e \in \{e_1, e_2, e_3\}\) and \(f \in \{f_1, f_2, f_3\}\) be arbitrary. What are the coadjoint orbits?

\(\text{SE}(3) \cdot (0, 0) = (0, 0)\). Since \(\text{SE}(3)_{(0, 0)} = \text{SE}(3)\) is not compact, the coadjoint action is not proper.
The orbit through \((e, 0), \ e \neq 0,\) is

\[ \text{SE}(3) \cdot (e, 0) = \{ (Ae, 0) \mid A \in \text{SO}(3) \} = S_{||e||}^2 \times \{0\}, \]

the two-sphere of radius \(||e||\).

The orbit through \((0, f), \ f \neq 0,\) is

\[ \text{SE}(3) \cdot (0, f) = \{ (a \times Af, Af) \mid A \in \text{SO}(3), \ a \in \mathbb{R}^3 \} \]
\[ = \{ (u, Af) \mid A \in \text{SO}(3), \ u \perp Af \} = T S_{||f||}^2, \]

the tangent bundle of the two-sphere of radius \(||f||\); note that the vector part is the first component. We can think of it also as \(T^* S_{||f||}^2.\)
The orbit through $(e,f)$, where $e \neq 0, f \neq 0$, equals

$$\text{SE}(3) \cdot (e,f) = \{ (Ae + a \times Af, Af) \mid A \in \text{SO}(3), a \in \mathbb{R}^3 \}.$$ 

To get a better description of this orbit, consider the smooth map

$$\varphi : (A,a) \in \text{SE}(3) \mapsto \left( Ae + a \times Af - \frac{e \cdot f}{\|f\|^2} Af, Af \right) \in T\mathbb{S}^2\|f\|,$$

which is right invariant under the isotropy group

$$\text{SE}(3)_{(e,f)} = \{ (B,b) \mid Be + b \times f = e, Bf = f \}$$

and induces hence a diffeomorphism $\bar{\varphi} : \text{SE}(3)/\text{SE}(3)_{(e,f)} \to T\mathbb{S}^2\|f\|$. 
The orbit through \((e, f)\) is diffeomorphic to \(SE(3)/\ SE(3)_{(e,f)}\) by the diffeomorphism

\[(A, a) \mapsto Ad_{(A,a)}^{-1}(e, f).\]

Composing these two maps and identifying \(TS^2\) and \(T^*S^2\) by the natural Riemannian metric on \(S^2\), we get the diffeomorphism \(\Phi : SE(3) \cdot (e, f) \to T^*S^2_{\|f\|}\) given by

\[
\Phi(Ad_{(A,a)}^{-1}(e, f)) = \left( Ae + a \times Af - \frac{e \cdot f}{\|f\|^2} Af, Af \right).
\]

Thus this orbit is also diffeomorphic to \(T^*S^2_{\|f\|}\).
• SE(3) acting on $\mathbb{R}^3$

This action is proper: $(A, a) \cdot u := Au + a$. It is not a representation. The orbit through the origin is $\mathbb{R}^3$, $SE(3)_0 = SO(3)$.

This action is transitive: given $u \in \mathbb{R}^3$ we have $(I, 0) \cdot u = u$. So there is only one single orbit which is $\mathbb{R}^3$. 
A **symplectic manifold** is a pair \((M, \omega)\), where \(M\) is a manifold and \(\omega \in \Omega^2(M)\) is a closed non–degenerate two–form on \(M\), that is,

- \(d\omega = 0\)

- for every \(m \in M\), the map

\[v \in T_m M \mapsto \omega(m)(v, \cdot) \in T^*_m M\]

is a linear isomorphism.
If \( \omega \) is allowed to be degenerate, \((M, \omega)\) is called a **presymplectic manifold**. A **Hamiltonian dynamical system** is a triple \((M, \omega, h)\), where \((M, \omega)\) is a symplectic manifold and \(h \in C^\infty(M)\) is the **Hamiltonian function** of the system. By non–degeneracy of the symplectic form \(\omega\), to each Hamiltonian system one can associate a **Hamiltonian vector field** \(X_h \in \mathfrak{X}(M)\), defined by the equality

\[
\mathbf{i}_{X_h}\omega = dh.
\]
Example  Let $V$ be a vector space and $V^*$ its dual. Let $Z = V \times V^*$. The canonical symplectic form $\Omega$ on $Z$ is defined by

$$\Omega(((v_1, \alpha_1), (v_2, \alpha_2)) := \langle \alpha_2, v_1 \rangle - \langle \alpha_1, v_2 \rangle.$$  

Example  Let $Q$ be a smooth manifold and $T^*Q$ its cotangent bundle. Let $\pi_Q : T^*Q \to Q$ be the projection and $\Theta$ the one–form on $T^*Q$ defined by

$$\Theta(\beta) \cdot v_\beta := \langle \beta, T_\beta \pi_Q (v_\beta) \rangle,$$

where $\beta \in T^*Q$ and $v_\beta \in T_\beta(T^*Q)$. The canonical symplectic form $\Omega$ on the cotangent bundle $T^*Q$ is defined by $\Omega = -d\Theta$.  

Darboux theorem: Locally $\omega|_U = \sum_{i=1}^{n} dq^i \wedge dp_i$.

In canonical coordinates, $X_h$ is determined by the well-known Hamilton equations,

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i}.$$ 

The Poisson bracket of $f, g \in C^\infty(M)$ is the function $\{f, g\} \in C^\infty(M)$ defined by

$$\{f, g\}(z) = \omega(z)(X_f(z), X_g(z)).$$
In canonical coordinates, the Poisson bracket takes the form

\[
\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right).
\]
POISSON MANIFOLDS

• \((M, \{\cdot, \cdot\})\) Poisson manifold if \((C^\infty(M), \{\cdot, \cdot\})\) Lie algebra such that

\[
\{fg, h\} = f\{g, h\} + g\{f, h\}
\]

• Casimir functions are the elements of the center of \((C^\infty(M), \{\cdot, \cdot\})\).

• Hamiltonian vector field of \(h \in C^\infty(M)\)

\[
X_h[f] = \{f, h\}, \quad \text{for all} \quad f \in C^\infty(M).
\]
Example: The Lie-Poisson bracket. The dual $g^*$ of a Lie algebra $g$ is a Poisson manifold with respect to the $\pm$-Lie–Poisson brackets $\{\cdot, \cdot\}_\pm$ defined by

$$\{f, g\}_\pm(\mu) := \pm \left\langle \mu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle,$$

where $\frac{\delta f}{\delta \mu} \in g$ is defined by

$$\left\langle \nu, \frac{\delta f}{\delta \mu} \right\rangle := Df(\mu) \cdot \nu,$$

for any $\nu \in g^*$. The Hamiltonian vector field of $h \in C^\infty(g^*)$ is given by

$$X_h(\mu) = \mp \text{ad}^*_{\delta h/\delta \mu} \mu, \quad \mu \in g^*.$$
Example: Frozen Lie-Poisson bracket. Same notations as before. Let $\nu \in g^*$ and define the frozen Lie–Poisson brackets $\{ \cdot, \cdot \}_\pm$ defined by

$$\{ f, g \}^\nu_\pm(\mu) := \pm \left\langle \nu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle.$$  

The Hamiltonian vector field of $h \in C^\infty(g^*)$ is given by

$$X_h(\mu) = \mp \text{ad}_{\delta h/\delta \mu}^* \nu, \quad \mu \in g^*.$$  

The Lie-Poisson and frozen Lie-Poisson bracket are compatible, that is, $\{ \cdot, \cdot \}_\pm + s\{ \cdot, \cdot \}^\nu_\pm$ is also a Poisson bracket on $g^*$ for any $\nu \in g^*$ and any $s \in \mathbb{R}$.  

Example: Operator Algebra Brackets. \( \mathcal{H} \) be a complex Hilbert space.

- \( \mathcal{S}(\mathcal{H}) \), trace class operators
- \( \mathcal{HS}(\mathcal{H}) \), Hilbert-Schmidt operators
- \( \mathcal{K}(\mathcal{H}) \), compact operators
- \( \mathcal{B}(\mathcal{H}) \), bounded operators

They form involutive Banach algebras. \( \mathcal{S}(\mathcal{H}), \mathcal{HS}(\mathcal{H}), \mathcal{K}(\mathcal{H}) \) are self adjoint ideals in \( \mathcal{B}(\mathcal{H}) \).
\( \mathcal{S}(\mathcal{H}) \subset \mathcal{HS}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}) \)

\( \mathcal{K}(\mathcal{H})^* \cong \mathcal{S}(\mathcal{H}), \quad \mathcal{HS}(\mathcal{H})^* \cong \mathcal{HS}(\mathcal{H}), \quad \mathcal{S}(\mathcal{H})^* \cong \mathcal{B}(\mathcal{H}); \)

the right hand sides are all Banach Lie algebras. These dualities are implemented by the strongly nondegenerate pairing

\[ \langle x, \rho \rangle = \text{trace} (x\rho) \]

where \( x \in \mathcal{S}(\mathcal{H}), \ \rho \in \mathcal{K}(\mathcal{H}) \) for the first isomorphism, \( \rho, x \in \mathcal{HS}(\mathcal{H}) \) for the second isomorphism, and \( x \in \mathcal{B}(\mathcal{H}), \ \rho \in \mathcal{S}(\mathcal{H}) \) for the third isomorphism.
The Banach spaces $\mathcal{S}(\mathcal{H})$, $\mathcal{H}\mathcal{S}(\mathcal{H})$, and $\mathcal{K}(\mathcal{H})$ are Banach Lie-Poisson spaces in a rigorous functional analytic sense. The Lie-Poisson bracket becomes in this case

$$\{F, H\}(\rho) = \pm \text{trace} ([D F(\rho), D H(\rho)] \rho)$$

where $\rho$ is an element of $\mathcal{S}(\mathcal{H})$, $\mathcal{H}\mathcal{S}(\mathcal{H})$, or $\mathcal{K}(\mathcal{H})$, respectively. The bracket $[D F(\rho), D H(\rho)]$ denotes the commutator bracket of operators. The Hamiltonian vector field associated to $H$ is given by

$$X_H(\rho) = \pm[D H(\rho), \rho].$$
**The Poisson tensor.** The derivation property of the Poisson bracket implies that for any two functions $f, g \in C^\infty(M)$, the value of the bracket $\{f, g\}(z)$ on $f$ only through $df(z)$ which allows us to define a contravariant antisymmetric two-tensor $B \in \Lambda^2(M)$ by

$$B(z)(\alpha_z, \beta_z) = \{f, g\}(z),$$

with $df(z) = \alpha_z$ and $dg(z) = \beta_z$. This tensor is called the **Poisson tensor** of $M$. The vector bundle map $B^\#: T^*M \to TM$ naturally associated to $B$ is defined by

$$B(z)(\alpha_z, \beta_z) = \langle \alpha_z, B^\#(\beta_z) \rangle.$$
Its range $D := B^\#(T^*M) \subset TM$ is called the **characteristic distribution**. For any point $m \in M$, the dimension of $D(m)$ as a vector subspace of $T_mM$ is called the **rank** of the Poisson manifold $(M, \{\cdot, \cdot\})$ at the point $m$. 
The Weinstein coordinates of a Poisson manifold.

Let \((M, \{\cdot, \cdot\})\) be a \(m\)--dimensional Poisson manifold and \(z_0 \in M\) a point where the rank of \((M, \{\cdot, \cdot\})\) equals \(2n\), \(0 \leq 2n \leq m\). There exists a chart \((U, \varphi)\) of \(M\) whose domain contains the point \(z_0\) and such that the associated local coordinates, denoted by

\[
(q^1, \ldots, q^n, p_1, \ldots, p_n, z^1, \ldots, z^{m-2n}),
\]

satisfy

\[
\{q^i, q^j\} = \{p_i, p_j\} = \{q^i, z^k\} = \{p_i, z^k\} = 0,
\]

and \(\{q^i, p_j\} = \delta^i_j\), for all \(i, j, k, 1 \leq i, j \leq n, 1 \leq k \leq m-2n\).
For all $k, l, 1 \leq k, l \leq m - 2n$, the Poisson bracket $\{z^k, z^l\}$ is a function of the local coordinates $z^1, \ldots, z^{m-2n}$ exclusively, and vanishes at $z_0$. Hence, the restriction of the bracket $\{\cdot, \cdot\}$ to the coordinates $z^1, \ldots, z^{m-2n}$ induces a Poisson structure that is usually referred to as the transverse Poisson structure of $(M, \{\cdot, \cdot\})$ at $m$.

If the rank is equal to $2n$ in a neighborhood of $z_0$, then the transverse structure is zero.
A smooth mapping $\varphi : (M_1, \{\cdot,\cdot\}_1) \to (M_2, \{\cdot,\cdot\}_2)$ is **canonical** or **Poisson** if for all $g, h \in C^\infty(M_2)$ we have

$$\varphi^*\{g, h\}_2 = \{\varphi^*g, \varphi^*g\}_1.$$  

In the symplectic category, $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ **canonical** or **symplectic** if

$$\varphi^*\omega_2 = \omega_1.$$  

- Symplectic maps are immersions.
• A diffeomorphism $\varphi : M_1 \to M_2$ between two symplectic manifolds $(M_1, \omega_1)$ and $(M_2, \omega_2)$ is symplectic if and only if it is Poisson.

• If the symplectic map $\varphi : M_1 \to M_2$ is not a diffeomorphism it may not be a Poisson map.

Let $(S, \{\cdot, \cdot\}^S)$ and $(M, \{\cdot, \cdot\}^M)$ be two Poisson manifolds such that $S \subset M$ and the inclusion $i_S : S \hookrightarrow M$ is an immersion. $(S, \{\cdot, \cdot\}^S)$ is a Poisson submanifold of $(M, \{\cdot, \cdot\}^M)$ if $i_S$ is a canonical map.
An immersed submanifold \( Q \) of \( M \) is called a **quasi Poisson submanifold** of \( (M, \{\cdot, \cdot\}^M) \) if for any \( q \in Q \), any open neighborhood \( U \) of \( q \) in \( M \), and any \( f \in C^\infty_M(U) \) we have

\[
X_f(i_Q(q)) \in T_q i_Q(T_qQ),
\]

where \( i_Q : Q \hookrightarrow M \) is the inclusion and \( X_f \) is the Hamiltonian vector field of \( f \) on \( U \) with respect to the restricted Poisson bracket \( \{\cdot, \cdot\}^M_U \).

- On a quasi Poisson submanifold there is a unique Poisson structure that makes it into a Poisson submanifold.

- Any Poisson submanifold is quasi Poisson.
The converse is not true!

**Counterexample.** Let \((M = \mathbb{R}^2, B)\) where

\[
B(x, y) = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}
\]

and \((Q = \mathbb{R}^2, \omega_{\text{can}})\). The identity map \(id : Q \to M\) is obviously not a Poisson diffeomorphism because one structure has leaves and the other is non-degenerate. But it is also clear that any Hamiltonian vector field relative to \(B\) is tangent to \(Q = \mathbb{R}^2\) and hence \((Q, \omega_{\text{can}})\) is a quasi-Poisson submanifold of \((M, B)\).
Given two symplectic manifolds \((M, \omega)\) and \((S, \omega_S)\) such that \(S \subset M\) and the inclusion \(i : S \hookrightarrow M\) is an immersion, the manifold \((S, \omega_S)\) is a **symplectic submanifold** of \((M, \omega)\) when \(i\) is a symplectic map.

Symplectic submanifolds of a symplectic manifold \((M, \omega)\) are in general neither Poisson nor quasi Poisson manifolds of \(M\).

The only quasi Poisson submanifolds of a symplectic manifold are its open sets which are, in fact, Poisson submanifolds.
Symplectic Foliation Theorem. Let \((M, \{\cdot, \cdot\})\) be a Poisson manifold and \(D\) the associated characteristic distribution. \(D\) is a smooth and integrable generalized distribution and its maximal integral leaves form a generalized foliation decomposing \(M\) into initial submanifolds \(\mathcal{L}\), each of which is symplectic with the unique symplectic form that makes the inclusion \(i : \mathcal{L} \hookrightarrow M\) into a Poisson map, that is, \(\mathcal{L}\) is a Poisson submanifold of \((M, \{\cdot, \cdot\})\).
**Example:** Let \( g^* \) with the Lie-Poisson structure. The symplectic leaves of the Poisson manifolds \((g^*, \{\cdot, \cdot\}_\pm)\) coincide with the connected components of the orbits of the elements in \( g^* \) under the coadjoint action. In this situation, the symplectic form for the leaves is given by the **Kostant–Kirillov–Souriau (KKS) or orbit symplectic form**

\[
\omega^\pm_O(\nu)(\xi_{g^*}(\nu), \eta_{g^*}(\nu)) = \pm \langle \nu, [\xi, \eta]\rangle.
\]
• $(M, \{\cdot, \cdot\})$ Poisson manifold. $G$ acts **canonically** on $M$ when

$$\Phi^*_g \{f, h\} = \{\Phi^*_g f, \Phi^*_g h\}$$

for all $g \in G$.

• **Easy Poisson reduction:** $(M, \{\cdot, \cdot\})$ Poisson manifold, $G$ Lie group acting canonically, freely, and properly on $M$. The orbit space $M/G$ is a Poisson manifold with bracket

$$\{f, g\}^{M/G}(\pi(m)) = \{f \circ \pi, g \circ \pi\}(m)$$
• **Reduction of Hamiltonian dynamics:** \( h \in C^\infty(M)^G \) reduces to \( \bar{h} \in C^\infty(M/G) \) given by \( \bar{h} \circ \pi = h \) such that

\[
X_{\bar{h}} \circ \pi = T\pi \circ X_h
\]

• What about the symplectic leaves? This is where symplectic reduction comes in.

• **Lie-Poisson reduction:** Left quotient \( (T^*G)/G \cong g^* \). The map is: \([\alpha_g] \mapsto T_e^*R_g(\alpha_g)\).
Note: The displayed content is not clearly legible due to the quality of the image. Here is a transcription of the visible content:

- Consider $\mathbb{R}^6$ with the bracket

  $$\{f, g\} = \sum_{i=1}^{3} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right)$$

- $S^1$-action given by

  $$\Phi : S^1 \times \mathbb{R}^6 \rightarrow \mathbb{R}^6$$

  $$(e^{i\phi}, (x, y)) \mapsto (R_\phi x, R_\phi y)$$
• Hamiltonian of the spherical pendulum

\[ h = \frac{1}{2} \langle y, y \rangle + \langle x, e_3 \rangle \]

• Impose constraint \( \langle x, x \rangle = 1 \)

• Angular momentum: \( J(x, y) = x_1 y_2 - x_2 y_1 \).
Hilbert-Weyl Theorem: $H \to \text{Aut}(V)$ representation, $H$ compact Lie group. Then the algebra $\mathcal{P}(V)^H$ of $H$-invariant polynomials on $V$ is finitely generated, i.e., \( \forall P \in \mathcal{P}(V)^H, \exists k \in \mathbb{N}, \pi_1, \ldots, \pi_k \in \mathcal{P}(V)^H, \hat{P} \in \mathbb{R}[X_1, \ldots, X_k] \) s.t. \( P = \hat{P} \circ (\pi_1, \ldots, \pi_k) \). Minimal set is a Hilbert basis.

Hilbert basis of the algebra of $S^1$-invariant polynomials on $\mathbb{R}^6$ is given by

- $\sigma_1 = x_3$
- $\sigma_2 = y_3$
- $\sigma_3 = y_1^2 + y_2^2 + y_3^2$
- $\sigma_4 = x_1y_1 + x_2y_2$
- $\sigma_5 = x_1^2 + x_2^2$
- $\sigma_6 = x_1y_2 - x_2y_1$.

Semialgebraic relations

\[ \sigma_4^2 + \sigma_6^2 = \sigma_5(\sigma_3 - \sigma_2^2), \quad \sigma_3 \geq 0, \quad \sigma_5 \geq 0. \]
Hilbert map $\pi : v \in V \mapsto (\pi_1(v), \ldots, \pi_k(v)) \in \mathbb{R}^k$ separates $H$-orbits. So $V/H \cong \text{range}(\pi)$.

**Schwarz Theorem:** The map $f \in C^\infty(\mathbb{R}^k) \mapsto f \circ (\pi_1, \ldots, \pi_k) \in C^\infty(V)^H$ is surjective.

**Mather Theorem:** The quotient presheaf of smooth functions on $V/H$ is isomorphic to the presheaf of Whitney smooth functions on $\pi(V)$ induced by the sheaf of smooth functions on $\mathbb{R}^k$.

**Tarski-Seidenberg Theorem:** Since $\pi$ is a polynomial map, $\text{range}(\pi) \subset \mathbb{R}^k$ is semi-algebraic.
**Theorem:** Every semi-algebraic set admits a canonical Whitney stratification into a finite number of semi-algebraic subsets.

**Bierstone Theorem:** This canonical stratification of $\pi(V)$ coincides with the stratification of $V/H$ into orbit type manifolds.

These theorems can be used to explicitly describe quotient spaces of representations as semi-algebraic subsets of a (high dimensional) Euclidean space.

Return to our concrete case of the spherical pendulum.
The Hilbert map is given by

\[ \sigma : \mathbb{T} \mathbb{R}^3 \rightarrow \mathbb{R}^6 \]
\[ (x, y) \mapsto (\sigma_1(x, y), \ldots, \sigma_6(x, y)) \].

The \( S^1 \)-orbit space \( \mathbb{T} \mathbb{R}^3 / S^1 \) can be identified with the semialgebraic variety \( \sigma(\mathbb{T} \mathbb{R}^3) \subset \mathbb{R}^6 \), defined by these relations.

\( TS^2 \) is a submanifold of \( \mathbb{R}^6 \) given by

\[ TS^2 = \{(x, y) \in \mathbb{R}^6 \mid \langle x, x \rangle = 1, \langle x, y \rangle = 0 \} \].

\( TS^2 \) is \( S^1 \)-invariant.
\(TS^2/S^1\) can be thought of the semialgebraic variety \(\sigma(TS^2)\) defined by the previous relations and

\[
\sigma_5 + \sigma_1^2 = 1 \quad \sigma_4 + \sigma_1 \sigma_2 = 0,
\]

which allow us to solve for \(\sigma_4\) and \(\sigma_5\), yielding

\[
TS^2/S^1 = \sigma(TS^2) = \{(\sigma_1, \sigma_2, \sigma_3, \sigma_6) \in \mathbb{R}^4 | \sigma_1^2 \sigma_2 + \sigma_6^2 = (1 - \sigma_1^2)(\sigma_3 - \sigma_2^2), \quad |\sigma_1| \leq 1, \sigma_3 \geq 0\}.
\]
The Poisson bracket is

\[ \{.,.\}_{TS^2/S^1} \]

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<tr>
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<th>( \sigma_1 )</th>
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<td>( 1 - \sigma_1^2 )</td>
<td>2( \sigma_2 )</td>
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<td>( \sigma_2 )</td>
<td>( -(1 - \sigma_1^2) )</td>
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<td>( -2\sigma_1\sigma_3 )</td>
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<td>( \sigma_3 )</td>
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The reduced Hamiltonian is

\[ H = \frac{1}{2} \sigma_3 + \sigma_1 \]
If $\mu \neq 0$ then $(TS^2)_\mu := J^{-1}(\mu)/S^1$ appears as the graph of the smooth function

$$\sigma_3 = \frac{\sigma_2^2 + \mu^2}{1 - \sigma_1^2}, \quad |\sigma_1| < 1.$$ 

The case $\mu = 0$ is singular and $(TS^2)_0 := J^{-1}(0)/S^1$ is not a smooth manifold.
ABSTRACT SYMMETRY REDUCTION

The case of general vector fields

$M$ manifold

$G \times M \to M$ smooth proper Lie group action

$X \in \mathfrak{X}(M)^G$, $G$-equivariant vector field

$F_t$ flow of $X \in \mathfrak{X}(M)^G$
Law of conservation of isotropy:

\[ M_H := \{ m \in M \mid G_m = H \} \], the \textit{H-isotropy type submanifold}, is preserved by \( F_t \).

\( M_H \) is, in general, not closed in \( M \).

Properness of the action implies:

- \( G_m \) is compact
- the (connected components of) \( M_H \) are embedded submanifolds of \( M \)
\(N(H)/H\) (where \(N(H)\) denotes the normalizer of \(H\) in \(G\)) acts freely and properly on \(M_H\).

\[\pi_H : M_H \to M_H/(N(H)/H)\] projection

\[i_H : M_H \hookrightarrow M\] inclusion

\(X\) induces a unique \(H\)-isotropy type reduced vector field \(X^H\) on \(M_H/(N(H)/H)\) by

\[X^H \circ \pi_H = T\pi_H \circ X \circ i_H,\]
whose flow $F_t^H$ is given by

$$F_t^H \circ \pi_H = \pi_H \circ F_t \circ i_H.$$ 

If $G$ is compact and the action is linear, then the construction of $M_H/(N(H)/H)$ can be implemented in a very explicit and convenient manner by using the invariant polynomials of the action and the theorems of Hilbert and Schwarz-Mather.
The Hamiltonian case

\((M, \omega)\) symplectic manifold, \(G\) connected Lie group with Lie algebra \(\mathfrak{g}\), \(G \times M \to M\) free proper symplectic action

\(J : M \to \mathfrak{g}^*\) equivariant **momentum map**:

- \(X_{J\xi} = \xi_M\)
- \(J^\xi := \langle J, \xi \rangle\)
- \(\xi_M\) infinitesimal generator given by \(\xi \in \mathfrak{g}\)
Noether’s Theorem: The fibers of $J$ are preserved by the Hamiltonian flows associated to $G$-invariant Hamiltonians. Equivalently, $J$ is conserved along the flow of any $G$-invariant Hamiltonian.

**Proof** Let $h \in C^\infty(M)$ be $G$-invariant, so $h \circ \Phi_g = h$ for any $g \in G$. Take the derivative of this relation at $g = e$ and get $\mathcal{L}_{\xi_M} h = 0$. But $\xi_M = X_{J^{-1}}$ so we get $\{J^{\xi}, h\} = \langle dh, X_{J^{\xi}} \rangle = \mathcal{L}_{\xi_M} h = 0$, which shows that $J^{\xi} \in C^\infty(M)$ is constant on the flow of $X_h$ for any $\xi \in \mathfrak{g}$, that is $J$ is conserved. $\square$
Example: lifted actions on cotangent bundles. Let $G$ be a Lie group acting on the manifold $Q$ and then by lift on its cotangent bundle $T^*Q$.

$$\langle J(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle,$$

for any $\alpha_q \in T^*Q$ and any $\xi \in g$.

Example: linear momentum. Take the phase space of the $N$–particle system, that is, $T^*\mathbb{R}^{3N}$. The additive group $\mathbb{R}^3$ acts on it by

$$v \cdot (q_i, p^i) = (q_i + v, p^i)$$
\[ \mathbf{J} : T^* \mathbb{R}^{3N} \longrightarrow \text{Lie}(\mathbb{R}^3) \cong \mathbb{R}^3 \]
\[ (q_i, p^i) \longmapsto \sum_{i=1}^{N} p_i \]
which is the classical **linear momentum**.

Let’s do this using the formula above.

\[ \langle \mathbf{J}(q_i, p^i), \xi \rangle = \sum_{i=1}^{N} p^i \cdot \xi \]
since \( \xi_{\mathbb{R}^3}(q_i) = (q_1, \ldots, q_N; \xi, \ldots \xi) \).

**Example: angular momentum.** Let \( \text{SO}(3) \) act on \( \mathbb{R}^3 \) and then, by lift, on \( T^* \mathbb{R}^3 \), that is, \( A \cdot (q, p) = (Aq, Ap) \).

\[ \mathbf{J} : T^* \mathbb{R}^3 \longrightarrow \mathfrak{so}(3)^* \cong \mathbb{R}^3 \]
\[ (q, p) \longmapsto q \times p. \]
which is the classical **angular momentum**.
Let’s do it using the formula for cotangent lifted actions. If \( \xi \in \mathbb{R}^3 \), \( \hat{\xi}v := \xi \times v \), for any \( v \in \mathbb{R}^3 \), \( \hat{\xi} \in so(3) \), then
\[
\xi_{\mathbb{R}^3}(v) = \frac{d}{dt} \bigg|_{t=0} e^{t\hat{\xi}}v = \hat{\xi}v = \xi \times v
\]
so that
\[
\left\langle J(q, p), \xi \right\rangle = p \cdot \xi_{\mathbb{R}^3}(q) = p \cdot (\xi \times q) = (q \times p) \cdot \xi
\]
which shows that
\[
J(q, p) = q \times p
\]
Example: symplectic linear actions. Let \((V, \omega)\) be a symplectic linear space and let \(G\) be a subgroup of the linear symplectic group, acting naturally on \(V\).

\[
\langle J(v), \xi \rangle = \frac{1}{2} \omega(\xi V(v), v).
\]

Example: Cayley-Klein parameters and the Hopf fibration. Consider the natural action of \(SU(2)\) on \(\mathbb{C}^2\). Since this action is by isometries of the Hermitian metric, it is automatically symplectic and therefore has a momentum map \(J : \mathbb{C}^2 \rightarrow \mathfrak{su}(2)^*\) given, as above, by

\[
\langle J(z, w), \xi \rangle = \frac{1}{2} \omega(\xi(z, w)^T, (z, w)), \quad z, w \in \mathbb{C}, \ \xi \in \mathfrak{su}(2).
\]
The Lie algebra $\mathfrak{su}(2)$ of SU(2) consists of $2 \times 2$ skew Hermitian matrices of trace zero. This Lie algebra is isomorphic to $\mathfrak{so}(3)$ and therefore to $(\mathbb{R}^3, \times)$ by the isomorphism given by

$$x = (x^1, x^2, x^3) \in \mathbb{R}^3 \mapsto \tilde{x} := \frac{1}{2} \begin{bmatrix} -ix^3 & -ix^1 - x^2 \\ -ix^1 + x^2 & ix^3 \end{bmatrix} \in \mathfrak{su}(2).$$

Thus we have

$$[\tilde{x}, \tilde{y}] = (x \times y)\tilde{}, \quad \forall x, y \in \mathbb{R}^3.$$ 

Other useful formulas are

$$\det(2\tilde{x}) = \|x\|^2 \quad \text{and} \quad \text{trace}(\tilde{x}\tilde{y}) = -\frac{1}{2} x \cdot y.$$
Identify $\mathfrak{su}(2)^*$ with $\mathbb{R}^3$ by the map $\mu \in \mathfrak{su}(2)^* \mapsto \tilde{\mu} \in \mathbb{R}^3$ defined by

$$\tilde{\mu} \cdot x := -2\langle \mu, \tilde{x} \rangle$$

for any $x \in \mathbb{R}^3$.

The symplectic form on $\mathbb{C}^2$ is given by minus the imaginary part of the Hermitian inner product.
With these notations, the momentum map $\tilde{\mathcal{J}} : \mathbb{C}^2 \to \mathbb{R}^3$ can be explicitly computed in coordinates: for any $x \in \mathbb{R}^3$ we have

$$\tilde{\mathcal{J}}(z,w) \cdot x = -2 \langle \mathcal{J}(z,w), \tilde{x} \rangle$$

$$= \frac{1}{2} \text{Im} \left( \begin{bmatrix} -ix^3 & -ix^1 - x^2 \\ -ix^1 + x^2 & ix^3 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \cdot \begin{bmatrix} z \\ w \end{bmatrix} \right)$$

$$= -\frac{1}{2} (2 \text{Re}(w\bar{z}), 2 \text{Im}(w\bar{z}), |z|^2 - |w|^2) \cdot x.$$

Therefore

$$\tilde{\mathcal{J}}(z,w) = -\frac{1}{2} (2w\bar{z}, |z|^2 - |w|^2) \in \mathbb{R}^3.$$
$\mathcal{J}$ is a Poisson map from $\mathbb{C}^2$, endowed with the canonical symplectic structure, to $\mathbb{R}^3$, endowed with the $\bot$ Lie Poisson structure. Therefore, $-\mathcal{J} : \mathbb{C}^2 \rightarrow \mathbb{R}^3$ is a canonical map, if $\mathbb{R}^3$ has the $\bot$ Lie-Poisson bracket relative to which the free rigid body equations are Hamiltonian. Pulling back the Hamiltonian

$$H(\Pi) = \frac{1}{2} \Pi \cdot \mathbb{I}^{-1} \Pi,$$

$$\mathbb{I}^{-1} \Pi := \left( \frac{\Pi_1}{I_1}, \frac{\Pi_2}{I_2}, \frac{\Pi_3}{I_3} \right)$$

to $\mathbb{C}^2$ gives a Hamiltonian function (called collective) on $\mathbb{C}^2$. $\mathbb{I} = \text{diag}(I_1, I_2, I_3)$ is the moment of inertia tensor written in a principal axis body frame of the free rigid body.
The classical Hamilton equations for this function are therefore projected by $-\mathbf{\dot{J}}$ to the rigid body equations

$$\mathbf{\dot{\Pi}} = \mathbf{\Pi} \times \mathbf{I}^{-1}\mathbf{\Pi}.$$ 

In this context, the variables $(z, w)$ are called the **Cayley-Klein parameters**. They represent a first attempt to understand the rigid body equations as a Hamiltonian system, before the introduction of Poisson manifolds. In quantum mechanics, the same variables are called the **Kustaanheimo-Stiefel coordinates**. A similar construction was carried out in fluid dynamics making the Euler equations a Hamiltonian system relative to the so-called **Clebsch variables**.
Now notice that if

\[(z, w) \in S^3 := \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\},\]

then \[\| -\tilde{\mathbf{J}}(z, w)\| = 1/2,\] so that \[-\tilde{\mathbf{J}}|_{S^3} : S^3 \rightarrow S^2_{1/2},\] where \(S^2_{1/2}\) is the sphere in \(\mathbb{R}^3\) of radius \(1/2\).

It is also easy to see that \(-\tilde{\mathbf{J}}|_{S^3}\) is surjective and that its fibers are circles. Indeed, given \((x^1, x^2, x^3) = (x^1 + ix^2, x^3) = (re^{i\psi}, x^3) \in S^2_{1/2},\) the inverse image of this point is

\[-\tilde{\mathbf{J}}^{-1}(re^{i\psi}, x^3) = \left\{ \left( e^{i\theta} \sqrt{\frac{1}{2} + x^3}, e^{i\varphi} \sqrt{\frac{1}{2} - x^3} \right) \in S^3 \mid e^{i(\theta - \varphi + \psi)} = 1 \right\}.\]
One recognizes now that $-\tilde{\mathbf{J}}|_{S^3} : S^3 \rightarrow S^2_{1/2}$ is the **Hopf fibration**. In other words:

the momentum map of the SU(2)-action on $\mathbb{C}^2$, the Cayley-Klein parameters, the Kustaanheimo-Stiefel co-ordinates, and the family of Hopf fibrations on concentric three-spheres in $\mathbb{C}^2$ are the same map.
• Freeness of the action is equivalent to the regularity of the momentum map: \( \text{range} T_m J = (g_m) \circ \).

Proof: We have \( T_m M = \{ X_f(m) \mid f \in C^\infty(U) \} \), \( U \) open neighborhood of \( m \). For any \( \xi \in g \) we have

\[
\langle T_m J (X_f(m)), \xi \rangle = dJ^\xi(m) (X_f(m)) = \{ J^\xi, f \}(m) \\
= -d f(m) (X_{J^\xi}(m)) = -d f(m) (\xi_M(m)).
\]

So

\[
\xi \in g_m \iff \xi_M(m) = 0 \iff \\
d f(m) (\xi_M(m)) = 0, \forall f \in C^\infty(U) \iff \\
\langle T_m J (X_f(m)), \xi \rangle = 0, \forall f \in C^\infty(U) \iff \\
\xi \in (\text{range } T_m J)^\circ \quad \square
\]
• \( \ker T_m J = (g \cdot m)^\omega \).

Proof: \( v_m \in \ker T_m J \) if and only if for all \( \xi \in g \)

\[
0 = \langle T_m J(v_m), \xi \rangle = dJ^\xi(m)(v_m) = \omega(m)(X_{J^\xi}(m), v_m) \\
= \omega(m)(\xi_M(m), v_m) \\
\iff v_m \in (g \cdot m)^\omega \quad \square
\]

• **Existence:** The obstruction is the vanishing of the map

\[
\rho: g/[g, g] \longrightarrow H^1(M, \mathbb{R}) \\
[\xi] \quad \mapsto \quad [i_{\xi_M} \omega]
\]
• **Equivariance:** When is \((g, [\cdot, \cdot]) \rightarrow (C^\infty(M), \{\cdot, \cdot\})\) defined by \(\xi \mapsto J^\xi\), \(\xi \in g\), a Lie algebra homomorphism, that is, 

\[
J^{[\xi, \eta]} = \{J^\xi, J^\eta\}, \quad \xi, \eta \in g.
\]

Answer: if and only if 

\[
T_z J (\xi_M(z)) = - \text{ad}_\xi^* J(z),
\]

A momentum map that satisfies this relation in called **infinitesimally equivariant**.

Among all possible choices of momentum maps for a given action, there is at most one infinitesimally equivariant one.
Sufficient conditions: Assume $H^1(g; \mathbb{R}) = H^2(g; \mathbb{R}) = 0$. By the **Whitehead lemmas**, this is the case if $g$ is semisimple.

- $J$ is **$G$-equivariant** when

  $$\text{Ad}^*_{g^{-1}} \circ J = J \circ \Phi_g$$

- If $G$ is compact $J$ can be chosen $G$-equivariant

- If $G$ is connected then infinitesimal equivariance is equivalent to equivariance.
Define the **non-equivariance one-cocycle** associated to \( J \) as the map

\[
\sigma : G \longrightarrow g^* \\
g \longmapsto J(\Phi_g(z)) - \operatorname{Ad}^*_{g^{-1}}(J(z)).
\]

Suppose that \( M \) is connected. Then:

(i) The definition of \( \sigma \) does not depend on the choice of \( z \in M \).

(ii) The mapping \( \sigma \) is a \( g^* \)-valued one-cocycle on \( G \) with respect to the coadjoint representation of \( G \) on \( g^* \).
We define the **affine action** of $G$ on $g^*$ with cocycle $\sigma$ by

$$\Theta : G \times g^* \longrightarrow g^*
(g, \mu) \longmapsto \text{Ad}_{g^{-1}}^* \mu + \sigma(g).$$

$\Theta$ determines a left action of $G$ on $g^*$. The momentum map $J : M \rightarrow g^*$ is equivariant with respect to the symplectic action $\Phi$ on $M$ and the affine action $\Theta$ on $g^*$.

The affine orbits $O_\mu$ are also symplectic with $G$-invariant symplectic structure given by

$$\omega_{O_\mu}^\pm (\nu)(\xi g^* (\nu), \eta g^* (\nu)) = \pm \langle \nu, [\xi, \eta]\rangle \mp \Sigma(\xi, \eta),$$
where the \textbf{infinitesimal non-equivariance two-cocycle} $\Sigma \in Z^2(g, \mathbb{R})$ is given by

$$
\Sigma : \ g \times g \longrightarrow \mathbb{R} \\
(\xi, \eta) \longmapsto \Sigma(\xi, \eta) = d\hat{\sigma}_\eta(e) \cdot \xi,
$$

with $\hat{\sigma}_\eta : G \to \mathbb{R}$ defined by $\hat{\sigma}_\eta(g) = \langle \sigma(g), \eta \rangle$.

\textbf{Reduction Lemma:}

$$
g_{J(m)} \cdot m = g \cdot m \cap \ker T_m J = g \cdot m \cap (g \cdot m)^\omega.
$$

\textbf{Proof:} $\xi_M(m) \in g \cdot m \cap \ker T_m J \iff 0 = T_m J(\xi_M(m)) = -\text{ad}_{\xi}^* J(m) + \Sigma(\xi, \cdot) \iff \xi \in g_{J(m)} \quad \Box$
The geometry of the reduction lemma.
Momentum maps and isotropy type manifolds.

• \( m \in M \). Then \( M_{Gm} \) is a symplectic submanifold of \( M \).

**Proof:** By the Tube Theorem for proper actions, \( M_{Gm} \) is an embedded submanifold and \( T_zM_{Gm} = T_zM^{Gm} = (T_zM)^{Gm}, \forall z \in M_{Gm} \). To show that \( i^*\omega \) is a symplectic form, where \( i : M_{Gm} \hookrightarrow M \), it suffices to show that \((i^*\omega)(z)\) is nondegenerate on \( T_zM_{Gm} \), for all \( z \in M_{Gm} \).

\( H \) compact Lie group and \( (V,\omega) \) symplectic representation space. Then \( V^H \) is a symplectic subspace of \( V \).
Let $\langle \cdot, \cdot \rangle$ be a $H$-invariant inner product on $V$, possible by compactness of $H$ (average some inner product). Define $T : V \to V$ by $\langle u, v \rangle = \omega(u, T v)$ and note that it is a $H$-equivariant isomorphism. Therefore, $T(V^H) \subset V^H$. Assume that $u \in V^H$ satisfies $\omega(u, v) = 0, \forall v \in V^H$. But then $0 = \omega(u, T v) = \langle u, v \rangle, \forall v \in V^H$. Put here $v = u$ and then the positive definiteness of $\langle \cdot, \cdot \rangle$ implies that $u = 0$. □
Let $M^m_{G_m}$ be the connected component of $M_{G_m}$ containing $m$ and

$$N(G_m)^m := \{ n \in N(G_m) \mid n \cdot z \in M^m_{G_m} \text{ for all } z \in M^m_{G_m} \}.$$ 

$N(G_m)^m$ is a closed subgroup of $N(G_m)$ that contains the connected component of the identity. So it is also open and hence $\text{Lie}(N(G_m)^m) = \text{Lie}(N(G_m))$.

In addition, $(N(G_m)/G_m)^m = N(G_m)^m/G_m$ so that

$$\text{Lie} (N(G_m)^m/G_m) = \text{Lie} (N(G_m)/G_m).$$
• $L^m := N(G_m)^m/G_m$ acts freely properly and canonically on $M^m_{G_m}$ by $\Psi(nG_m, z) := n \cdot z$.

**Proof:** The map $\Psi$ is clearly well defined. It is easy to see it is a left action. It is also obvious that it is free. It is proper, because $N(G_m)^m$ is closed. Still need to show that it is canonical.

For any $l = nG_m \in L^m$ we have

$$\Psi^*_l(i^*\omega) = (i \circ \Psi_l)^*\omega = (\Phi_n \circ i)^*\omega = i^*\Phi^*_n\omega = i^*\omega.$$
The free proper canonical action of \( L^m := N(G_m)^m/G_m \) on \( M^m_{G_m} \) has a momentum map \( J_{Lm} : M^m_{G_m} \to (\text{Lie}(L^m))^* \) given by

\[
J_{Lm}(z) := \Lambda(J|_{M^m_{G_m}}(z) - J(m)), \quad z \in M^m_{G_m}.
\]

In this expression \( \Lambda : (g_m^o)^{G_m} \to (\text{Lie}(L^m))^* \) denotes the natural \( L^m \)-equivariant isomorphism given by

\[
\left\langle \Lambda(\beta), \frac{d}{dt} \bigg|_{t=0} \exp(t\xi)G_m \right\rangle = \left\langle \beta, \xi \right\rangle,
\]

for any \( \beta \in (g_m^o)^{G_m}, \xi \in \text{Lie}(N(G_m)^m) = \text{Lie}(N(G_m)) \).
The non-equivariance one-cocycle $\tau : M^m_{G_m} \to (\text{Lie}(L^m))^*$ of the momentum map $J_{Lm}$ is given by the map

$$\tau(l) = \Lambda(\sigma(n) + n \cdot J(m) - J(m)).$$
CONVEXITY

$J : M \to g^*$ coadjoint equivariant. $G, M$ compact. The intersection of the image of $J$ with a Weyl chamber is a compact and convex polytope. This polytope is referred to as the momentum polytope.

Delzant’s theorem proves that the symplectic toric manifolds are classified by their momentum polytopes. A Delzant polytope in $\mathbb{R}^n$ is a convex polytope that is also:

(i) Simple: there are $n$ edges meeting at each vertex.
(ii) **Rational:** the edges meeting at a vertex $p$ are of the form $p + tu_i$, $0 \leq t < \infty$, $u_i \in \mathbb{Z}^n$, $i \in \{1, \ldots, n\}$.

(iii) **Smooth:** the vectors $\{u_1, \ldots, u_n\}$ can be chosen to be an integral basis of $\mathbb{Z}^n$.

Delzant’s Theorem can be stated by saying that

\[
\{\text{symplectic toric manifolds}\} \quad \longrightarrow \quad \{\text{Delzant polytopes}\} \\
(M, \omega, \mathbb{T}^n, J : M \to \mathbb{R}^n) \quad \longmapsto \quad J(M)
\]

is a bijection.
Marsden-Weinstein Reduction Theorem

• $J: M \rightarrow g^*$ equivariant (not essential)

• $\mu \in J(M) \subset g^*$ regular value of $J$

• $G_\mu$-action on $J^{-1}(\mu)$ is free and proper, where $G_\mu := \{g \in G \mid \text{Ad}^*_g \mu = \mu\}$

then $(M_\mu := J^{-1}(\mu)/G_\mu, \omega_\mu)$ is symplectic:

$$\pi^*_\mu \omega_\mu = i^*_\mu \omega,$$

$i_\mu : J^{-1}(\mu) \hookrightarrow M$ inclusion,

$\pi_\mu : J^{-1}(\mu) \rightarrow J^{-1}(\mu)/G_\mu$ projection.
The flow $F_t$ of $X_h$, $h \in C^\infty(M)^G$, leaves the connected components of $J^{-1}(\mu)$ invariant and commutes with the $G$-action, so it induces a flow $F^\mu_t$ on $M_\mu$ by

$$\pi_\mu \circ F_t \circ i_\mu = F^\mu_t \circ \pi_\mu.$$

$F^\mu_t$ is Hamiltonian on $(M_\mu, \omega_\mu)$ for the reduced Hamiltonian $h_\mu \in C^\infty(M_\mu)$ given by

$$h_\mu \circ \pi_\mu = h \circ i_\mu.$$

Moreover, if $h, k \in C^\infty(M)^G$, then $\{h, k\}_\mu = \{h_\mu, k_\mu\}_{M_\mu}.$
**Proof:** Since $\pi_{\mu}$ is a surjective submersion, if $\omega_{\mu}$ exists, it is uniquely determined by the condition $\pi_{\mu}^*\omega_{\mu} = i_{\mu}^*\omega$. This relation also defines $\omega_{\mu}$ by:

$$\omega_{\mu}(\pi_{\mu}(z)) (T_z\pi_{\mu}(v), T_z\pi_{\mu}(w)) := \omega(z)(v, w),$$

for $z \in J^{-1}(\mu)$ and $v, w \in T_zJ^{-1}(\mu)$.

To see that this is a good definition of $\omega_{\mu}$, let

$$y = \Phi_g(z), \quad v' = T_z\Phi_g(v) \quad \text{and} \quad w' = T_z\Phi_g(w),$$

where $g \in G_{\mu}$. If, in addition $T_{g \cdot z}\pi_{\mu}(v'') = T_{g \cdot z}\pi_{\mu}(v') = T_z\pi_{\mu}(v)$ and $T_{g \cdot z}\pi_{\mu}(w'') = T_{g \cdot z}\pi_{\mu}(w') = T_z\pi_{\mu}(w)$, then $v'' = v' + \xi_M(g \cdot z)$ and $w'' = w' + \eta_M(g \cdot z)$ for some $\xi, \eta \in g_{\mu}$ and hence
\( \omega(y)(v'', w'') = \omega(y)(v', w') \) (by the reduction lemma)

\[
= \omega(\Phi_g(z))(T_z \Phi_g(v), T_z \Phi_g(w)) \\
= (\Phi^*_g \omega)(z)(v, w) \\
= \omega(z)(v, w) \quad \text{(action is symplectic)}.
\]

Thus \( \omega_\mu \) is well-defined. It is smooth since \( \pi^*_\mu \omega_\mu \) is smooth. Since \( d\omega = 0 \), we get

\[
\pi^*_\mu d\omega_\mu = d\pi^*_\mu \omega_\mu = d\iota^*_\mu \omega = \iota^*_\mu d\omega = 0.
\]

Since \( \pi_\mu \) is a surjective submersion, we conclude that \( d\omega_\mu = 0 \).
To prove nondegeneracy of $\omega_\mu$, suppose that

$$\omega_\mu(\pi_\mu(z))(T_z\pi_\mu(v), T_z\pi_\mu(w)) = 0$$

for all $w \in T_z(J^{-1}(\mu))$. This means that

$$\omega(z)(v, w) = 0 \quad \text{for all} \quad w \in T_z(J^{-1}(\mu)),$$

i.e., that $v \in (T_z(J^{-1}(\mu)))^\omega = T_z(G \cdot z)$ by the Reduction Lemma. Hence

$$v \in T_z(J^{-1}(\mu)) \cap T_z(G \cdot z) = T_z(G_\mu \cdot z)$$

so that $T_z\pi_\mu(v) = 0$, thus proving nondegeneracy of $\omega_\mu$. 
Let \( Y \in \mathfrak{X}(M_{\mu}) \) be the vector field whose flow is \( F^\mu_t \). Therefore, from \( \pi_{\mu} \circ F_t \circ i_{\mu} = F^\mu_t \circ \pi_{\mu} \) it follows

\[
T\pi_{\mu} \circ X_h = Y \circ T\pi_{\mu} \quad \text{on} \quad J^{-1}(\mu).
\]

Also, \( h_{\mu} \circ \pi_{\mu} = h \circ i_{\mu} \) implies that \( dh_{\mu} \circ T\pi_{\mu} = dh \) on \( J^{-1}(\mu) \). Therefore, on \( J^{-1}(\mu) \) we get

\[
\pi^*_\mu (i_Y \omega_{\mu}) = i_{X_h} \pi^*_\mu \omega_{\mu} = i_{X_h} i^*_\mu \omega = i^*_\mu (i_{X_h} \omega) = i^*_\mu dh
\]

\[
= d(h \circ i_{\mu}) = d(h_{\mu} \circ \pi_{\mu}) = \pi^*_\mu dh_{\mu}
\]

\[
= \pi^*_\mu \left( i_{X_{h_{\mu}}} \omega_{\mu} \right),
\]

so \( i_Y \omega_{\mu} = i_{X_{h_{\mu}}} \omega_{\mu} \) since \( \pi_{\mu} \) is a surjective submersion. Hence \( Y = X_{h_{\mu}} \) because \( \omega_{\mu} \) is nondegenerate.
Finally, for \( m \in J^{-1}(\mu) \) we have

\[
\{h_\mu, k_\mu\}_{M_\mu}(\pi_\mu(m)) = \omega_\mu(\pi_\mu(m)) \left( X_{h_\mu}(\pi_\mu(m)), X_{k_\mu}(\pi_\mu(m)) \right)
\]

\[
= \omega_\mu(\pi_\mu(m)) (T_m \pi_\mu(X_h(m)), T_m \pi_\mu(X_k(m)))
\]

\[
= (\pi_\mu^* \omega_\mu)(m) (X_h(m), X_k(m))
\]

\[
= (i_\mu^* \omega)(m) (X_h(m), X_k(m))
\]

\[
= \omega(m) (X_h(m), X_k(m))
\]

\[
= \{h, k\}(m)
\]

\[
= \{h, k\}_\mu(\pi_\mu(m)),
\]

which shows that \( \{h_\mu, k_\mu\}_{M_\mu} = \{h, k\}_\mu \). \qed
Problems with the reduction procedure

- Momentum map inexistent
- How does one recover the conservation of isotropy?
- $M_\mu$ is not a smooth manifold
- $G$ is discrete so momentum map is zero
- $M$ is not a symplectic but a Poisson manifold
Initial manifolds

$M$ and $N$ smooth manifolds, $N \subset M$ as sets. $N$ is called an **initial submanifold** of $M$ if the inclusion map $i : N \hookrightarrow M$ is an immersion satisfying the following condition: for any smooth manifold $P$ and any map $g : P \rightarrow N$, $g$ is smooth if and only if $i \circ g : P \rightarrow M$ is smooth. The smooth manifold structure that makes $N$ into an initial submanifold of $M$ is unique.
Generalized foliations

\[ \Phi = \{ \mathcal{L}_\alpha \}_{\alpha \in A} \] partition of \( M \) into disjoint connected sets, called \textit{leaves}, such that each point \( z \in M \) has a \textit{generalized foliated chart}:

- \((U, \varphi : U \to W \subset \mathbb{R}^m)\) with \( z \in U \)

- for each leaf \( \mathcal{L}_\alpha \) there is a natural number \( n \leq m \), called the \textit{dimension} of \( \mathcal{L}_\alpha \), and a subset \( A_\alpha \subset \mathbb{R}^{m-n} \) such that
\[ \varphi(U \cap \mathcal{L}_\alpha) = \{(z_1, \ldots, z_m) \in W \mid (z_{n+1}, \ldots, z_m) \in A_\alpha\}. \]

Each \((z_{n+1}^i, \ldots, z_m^i) \in A_\alpha\) determines a connected component \((U \cap \mathcal{L}_\alpha)^i\) of \(U \cap \mathcal{L}_\alpha\), that is, \(\varphi((U \cap \mathcal{L}_\alpha)^i) = \{(z_1, \ldots, z_n, z_{n+1}^i, \ldots, z_m^i) \in W\}\). Unlike in the case of standard foliations, the number \(n\) may change from leaf to leaf. The generalized foliated charts induce on the leaves a smooth manifold structure relative to which they are initial submanifolds of \(M\).
Generalized distributions

$D \subset TM$ such that, for any point $m \in M$, $D(m) := D \cap T_mM$ is a vector subspace of $T_mM$. The dimension of $D(m)$ is called the rank or the dimension of $D$ at the point $m$.

A differentiable section of $D$ is a differentiable vector field $X$ defined on an open subset $U$ of $M$, such that for any point $m \in U$, $X(m) \in D(m)$. 
An immersed connected submanifold $N$ of $M$ is said to be an **integral manifold** of $D$ if, for every $z \in N$, $T_zi(T_zN) \subset D(z)$, where $i : N \hookrightarrow M$ is the injection. The integral submanifold $N$ is said to be of **maximal dimension** at a point $z \in N$ if $T_zi(T_zN) = D(z)$.

$D$ is **differentiable** if, for every $m \in M$ and for every vector $v \in D(m)$, there exists a differentiable section $X$ of $D$, defined on an open neighborhood $U$ of $m$, such that $X(m) = v$. 
$D$ is **completely integrable** if, for every $m \in M$, there exists an integral manifold of $D$ everywhere of maximal dimension which contains $m$.

$D$ is **involutive** if it is invariant under the (local) flows associated to differentiable sections of $D$. This definition is more general than the usual one in the Frobenius theorem and it only coincides with it when the dimension of $D(m)$ is the same for any $m \in M$, that is, precisely when $D$ is a vector subbundle of $TM$.

**Stefan-Sussmann**: $D$ is completely integrable if and only if it is involutive.
$D$ integrable implies that for every $m \in M$ there exists a unique connected maximal integral manifold, or accessible set, $\mathcal{L}_m$ of $D$ that contains $m$ and which is maximal: it is everywhere of maximal dimension and if there is any other connected integral manifold $\mathcal{L}'$ of maximal dimension that intersects $\mathcal{L}_m$, then $\mathcal{L}'$ is an open submanifold of $\mathcal{L}_m$. The maximal integral manifolds of $D$ are initial submanifolds of $M$ and form a generalized foliation. Denote by $\Phi_D$ of $M$. $M/D \coloneqq M/\Phi_D$ the leaf space of $\Phi_D$. 
The term "accessible set" is justified by the fact that the maximal integral manifold $\mathcal{L}_m$ of $D$ containing the point $m$ coincides with the set of points that can be reached by applying to $m$ finite compositions of flows of the (locally defined) differentiable sections that span $D$. 
Pseudogroups

**Monoid**: set with an associative operation, two-sided identity (hence unique)

**Pseudogroup**: a submonoid $A$ of a given monoid such that each element has an inverse in $A$

**Example**: set of all local diffeomorphisms.

**Orbit** through $m \in M$:

\[ A \cdot m := \{ \varphi(m) \mid \varphi \in A, m \text{ in the domain of } \varphi \} \]
Orbits partition $M$

Projection $M \rightarrow M/A$ is continuous and open

$A$ is **integrable** if its orbits form a generalized foliation of $M$. Orbits are initial submanifolds.
**Generalized distribution** $D_A$ associated to $A$: $D_A(m) = T_m(A \cdot m)$ for all $m \in M$

$A$ has the **extension property** if any $f \in C^\infty(U)^A$, $U$ $A$-invariant open set, satisfies: for any $z \in U$, there is an $A$-invariant open neighborhood $V \subset U$ of $z$ and an $A$-invariant smooth function $F \in C^\infty(M)^A$ such that $f|_V = F|_V$.

The group of (global) diffeomorphisms of a proper Lie group action has the extension property.
Notation:

- $\mathcal{P}_L(M)$ pseudogroup of all local Poisson diffeomorphisms

- $\mathcal{P}(M)$ the group of Poisson diffeomorphisms
For $A \subset P_L(M)$, a pseudogroup, define:

- **Standard polar family** of $A$:

  $$F_A := \{X_f | f \in C^\infty(U)^A, U \subset M \text{ open and } A\text{-invariant}\}$$

- **Polar distribution** $D_{F_A}$ associated to $A$:

  $$D_{F_A}(m) := \{X_f(m) | f \in C^\infty(U)^A, U \subset M \text{ open and } A\text{-invariant}, m \in U\}$$

- **Polar family**: any generating family of vector fields for $D_{F_A}$
- **Polar pseudogroup** of $A$:

$$A' := \{ F_{t_1}^1 \circ \cdots \circ F_{t_n}^n \mid n \in \mathbb{N} \text{ and } F_{t_k}^k \text{ is a local flow of some } X_{f_k} \in F_A, 1 \leq k \leq n \}$$

**Example:** $A \subset \mathcal{P}_L(M)$ a pseudogroup of local Poisson diffeomorphisms with the extension property, then $\{ X_f \mid f \in C^\infty(M)^A \}$ is a polar family.
Properties of the polar distribution

Let $A \subset \mathcal{P}(M)$ be a group. Then:

(i) $A'$ acts canonically and is integrable.

(ii) Any $\psi \in A$ commutes with any $\varphi \in A'$.

(iii) Any $\varphi \in A'$ induces a local diffeomorphism $\bar{\varphi}$ of the presheaf space $(M/A, C^\infty_{M/A})$, that is,

$$\bar{\varphi} \circ \pi_A = \pi_A \circ \varphi,$$
where $\pi_A : M \to M/A$ is the projection. Hence $A'$ acts on the presheaf space $(M/A, C^\infty_{M/A})$.

(iiv) $A$ acts on $M/A'$, that is, for any $\psi \in A$, there is a diffeomorphism $\overline{\psi}$ of $(M/A', C^\infty_{M/A'})$ satisfying

$$\overline{\psi} \circ \pi_{A'} = \pi_{A'} \circ \psi,$$

where $\pi_{A'} : M \to M/A'$ is the projection.
The optimal momentum map

\[ \Phi : G \times M \to M \text{ canonical action on } (M, \{\cdot, \cdot\}) \]

\[ A_G := \{\Phi_g | g \in G\} \subset \mathcal{P}(M) \text{ a subgroup} \]

Standard polar family of \( A_G \):

\[ \{X_f | f \in C^\infty(U)^G, U \subset M \text{ open and } G\text{-invariant}\} \]

Polar pseudogroup or polar distribution of \( A_G \):

\[ A'_G = \{F_{t_1}^1 \circ \cdots \circ F_{t_n}^n | n \in \mathbb{N} \text{ and } F_{t_k}^k \text{ is a local flow of some } X_{f_k}, f_k \in C^\infty(U)^G, 1 \leq k \leq n, U \subset M \text{ open } G\text{-invariant}\} \]
The **optimal momentum map** \( J : M \rightarrow M/A'_G \) is, by definition, the projection (recall \( A'_G \) is integrable); it is continuous and open. \( J^{-1}(\rho) \) are initial submanifolds.

\( M/A'_G \) is called the **momentum space** of \( J \)

Proper \( G \)-action implies \( A_G \) has the extension property. Then \( J : M \rightarrow M/\mathcal{D}_F \) is the projection onto the leaf space of the integrable distribution spanned by the family of vector fields

\[
F := \left\{ X_f \mid f \in C^\infty(M)^G \right\}
\]

and the polar pseudogroup \( A'_G \) is a subgroup of the global diffeomorphisms group of \( M \).
Examples

1.) Poisson structure on $\mathbb{R}^3$

$$B = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}$$

$$X_f(x, y, z) = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \left( \frac{\partial f}{\partial z} - \frac{\partial f}{\partial x} \right) \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial}{\partial z}$$

has symplectic leaves $x + z = \text{constant}$ and Casimirs $f(x, y, z) := g(x + z)$, with $g \in C^\infty(\mathbb{R})$

$(\mathbb{R}, +)$ acts on $\mathbb{R}^3$ by $\lambda \cdot (x, y, z) := (x + \lambda, y, z)$
Action is canonical but does not admit a momentum map.

\[ f \in C^\infty(M)^\mathbb{R} \iff f(x, y, z) \equiv \bar{f}(y, z), \text{ with } \bar{f} \in C^\infty(\mathbb{R}^2) \text{ arbitrary.} \]

So \( A'_\mathbb{R} \)-orbits on \( \mathbb{R}^3 \) coincide with those of the \( \mathbb{R}^2 \)-action on \( \mathbb{R}^3 \) given by \((\mu, \nu) \cdot (x, y, z) := (x + \mu, y + \nu, z - \mu)\). Therefore, \( M/A'_G \cong \mathbb{R} \) and

\[ \mathcal{J}: (x, y, z) \in \mathbb{R}^3 \mapsto x + z \in \mathbb{R} \]
2.) $S^1 = \{e^{i\phi}\}$ acts canonically on 

\[(\mathbb{T}^2 = \{(e^{i\theta_1}, e^{i\theta_2})\}, d\theta_1 \wedge d\theta_2)\]

by $e^{i\phi} \cdot (e^{i\theta_1}, e^{i\theta_2}) := (e^{i(\theta_1+\phi)}, e^{i\theta_2})$ but does not admit a standard momentum map 

$$f \in C^\infty(\mathbb{T}^2)^{S^1} \Leftrightarrow f(e^{i\theta_1}, e^{i\theta_2}) = g(e^{i\theta_2})$$

for some $g \in C^\infty(S^1)$, so $X_f = \frac{\partial g}{\partial \theta_2} \frac{\partial}{\partial \theta_1}$.

Thus $M/A'_G \cong \{\text{the second circle } S^1 \text{ in the torus } \mathbb{T}^2\}$. So $\mathcal{J} : \mathbb{T}^2 \rightarrow S^1$ is given by

$$\mathcal{J}(e^{i\theta_1}, e^{i\theta_2}) = e^{i\theta_2}$$
3.) Canonical free circle action on

\[
(\mathbb{T}^2 \times \mathbb{T}^2, d\theta_1 \wedge d\theta_2 + \sqrt{2} d\psi_1 \wedge d\psi_2)
\]

\[
e^{i\phi} \cdot (e^{i\theta_1}, e^{i\theta_2}, e^{i\psi_1}, e^{i\psi_2})
\]

\[
:= (e^{i(\theta_1+\phi)}, e^{i\theta_2}, e^{i(\psi_1+\phi)}, e^{i\psi_2})
\]

Leaves of \(A'_{S_1}\) are dense in \(\mathbb{T}^2 \times \mathbb{T}^2\)
Know already that there is a smooth $G$-action on $M/A'_G$ (smooth in the sense of presheaf spaces) given by

$$g \cdot J(m) := J(g \cdot m).$$

This is the unique $G$-action on $M/A'_G$ that makes the optimal momentum map $G$-equivariant and it coincides with the usual smooth $G$-action on the leaf space of any distribution spanned by $G$-equivariant vector fields.

$J : M \rightarrow M/A'_G$ satisfies Noether’s condition: $J$ is constant on the flow of any Hamiltonian vector field of a $G$-invariant function.
Universality property: $\varphi$ canonical unique

\[
\begin{array}{ccc}
M & \xrightarrow{K} & P \\
\downarrow{\mathcal{J}} & & \downarrow{\varphi} \\
M/A_G & \rightarrow & \\
\end{array}
\]
$G$ acts properly and canonically on $(M, \omega)$. Then for any $m \in M$

$$A'_G(m) = (g \cdot m)^\omega \cap T_m M^m_{G_m}$$

$M^m_{G_m}$ is the connected component of the isotropy type submanifold $M_{G_m}$ that containing $m$.

$J : M \to g^*$ with **non-equivariance one-cocycle** $\sigma : G \to g^*$ given by

$$\sigma(g) := J(g \cdot m) - \text{Ad}^*_g J(m).$$

$\sigma$ does not depend on $m \in M$ if $M$ is connected.
$\Theta : G \times g^* \rightarrow g^*$ affine action

$$\Theta(g, \nu) := \text{Ad}^*_{g^{-1}} \nu + \sigma(g)$$

$G_\mu$ isotropy subgroup of $\mu$ with respect to $\Theta$.

If $J(m) = \mu$ and $J(m) = \rho$, we have

$$J^{-1}(\rho) = (J^{-1}(\mu) \cap M^m_{G_m})^m$$

$(J^{-1}(\mu) \cap M^m_{G_m})^m$ denotes the connected component of $J^{-1}(\mu) \cap M^m_{G_m}$ containing $m \in M$. 
\[ G_\rho = N_{G_\mu}(G_m)^{c(m)} \]

\( N_{G_\mu}(G_m)^{c(m)} \) is the closed subgroup of

\[ N_{G_\mu}(G_m) := N(G_m) \cap G_\mu \]

consisting of the elements in \( N_{G_\mu}(G_m) \) that leave \((J^{-1}(\mu) \cap M_m^m) \) invariant (as a set, not pointwise)

\( N(G_m) \) denotes the normalizer of \( G_m \) in \( G \).

\((M, \omega) \) connected, \( J : M \to g^* \) momentum map. Then for any \( m \in M \), the intersection \( J^{-1}(J(m)) \cap M_m^{m_Gm} \) is an embedded submanifold of \( M \).
Even though, in general, the level sets of the optimal momentum map are just initial submanifolds of $M$, this shows that in the symplectic case and if a standard momentum map exists, the level sets $\mathcal{J}^{-1}(\rho)$ are embedded submanifolds of $M$. 
THE OPTIMAL MOMENTUM MAP AND GROUPOIDS

Groupoids

$G \Rightarrow X$, $X$ is the base, $G$ is the total space

(i) $t, s : G \rightarrow X$; $t$ is the target, $s$ is the source

(ii) Set of composable pairs:

$$G^{(2)} := \{(g, h) \in G \times G \mid s(g) = t(h)\}.$$
The **product map** $m : G^{(2)} \to G$ satisfies

$$t(m(g, h)) = t(g)$$

$$s(m(g, h)) = s(h)$$

$$m(m(g, h), k) = m(g, m(h, k))$$

Write $gh := m(g, h)$.

(iii) **Identity section**: injection $\epsilon : X \to G$ such that

$$\epsilon(t(g))g = g = g\epsilon(s(g))$$. In particular, $t \circ \epsilon = s \circ \epsilon = \text{id}_X$. 
(iv) **Inversion map**: $i : G \rightarrow G$, also denoted by $i(g) = g^{-1}$, such that $g^{-1}g = \epsilon(s(g))$ and $gg^{-1} = \epsilon(t(g))$.

$G \Rightarrow X$, $H \subset G$ is a **subgroupoid** of $G$ if it is closed under multiplication and inversion. Then $H$ is a groupoid over $t(H) = s(H) \subset X$. If $t(H) = s(H) = X$, $H \Rightarrow X$ is called a **wide subgroupoid** of $G$.

**Lie groupoid**: all objects are smooth manifolds and smooth maps, the target and the source are surjective submersions. It follows that $\epsilon$ is an embedding and that $G^{(2)}$ is a submanifold of $G \times G$. 
Examples

1.) **Trivial groupoid**: \( X \) set, \( t = s = id_X \)

2.) **Pair groupoid**: \( X \) a set, \( G := X \times X \), \( t, s \) projections, \((x, y)(y, z) = (x, z)\), \( \epsilon(x) = (x, x) \), and \((x, y)^{-1} = (y, x)\)

3.) **A group**: \( G \) a group, \( X \) a one point set.

4.) **Product groupoid** \( G_1 \times G_2 \Rightarrow X_1 \times X_2 \)
5.) **Transformation groupoid**: $M \times \overline{A} \rightrightarrows M$, where $A$ is a pseudogroup of local diffeomorphisms of $M$ and

\[
\overline{A} = \{ \overline{\varphi} : M \to M \mid \varphi \in A, \overline{\varphi}(x) := \varphi(x) \text{ for } x \text{ in the domain of } \varphi \text{ and } \overline{\varphi}(x) := x \text{ if not} \}
\]

$t(x, \overline{\varphi}) = \overline{\varphi}(x)$, $s(x, \overline{\varphi}) = x$,
$m((x, \overline{\varphi}), (y, \overline{\psi})) := (y, \overline{\varphi} \circ \overline{\psi})$,
$\epsilon(x) = (x, id_M)$,
$(x, \overline{\varphi})^{-1} = (\overline{\varphi}(x), \overline{\varphi}^{-1})$, where $\overline{\varphi}, \overline{\psi} \in \overline{A}$. 
6.) **Action groupoid:** Special case for \( A := A_G := \{ \Phi_g \mid g \in G \} \), where \( \Phi : G \times M \to M \) is a smooth Lie group action.

\[
t(g, m) := g \cdot m, \quad s(g, m) := m,
\]
\[
\epsilon(m) := (e, m),
\]
\[
m((g, h \cdot n), (h, n)) := (gh, n),
\]
\[
(g, m)^{-1} := (g^{-1}, g \cdot m)
\]
7.) $T^*G$: Identify $T^*G$ with $G \times g^*$ using right translations.

t(g, \mu) := \text{Ad}^*_{g^{-1}} \mu, \ s(g, \mu) := \mu, \\
\epsilon(\mu) = (e, \mu), \\
m((g, \text{Ad}^*_{h^{-1}} \mu), (h, \mu)) = (gh, \mu), \\
(g, \mu)^{-1} = (g^{-1}, \text{Ad}^*_{g^{-1}} \mu).
8.) **The Baer groupoid:** \( \mathcal{G}(G) \) be the set of subgroups of \( G \)

\( \mathcal{B}(G) \) be the set of cosets of elements in \( \mathcal{G}(G) \)

\( \mathcal{B}(G) \rightrightarrows \mathcal{G}(G) \)

\[ t(D) = Dg^{-1}, \quad s(D) = g^{-1}D \]

\[(\mathcal{B}(G))^{(2)} := \{ (D_1, D_2) \in \mathcal{B}(G) \times \mathcal{B}(G) \mid \]

\[ g_1^{-1}D_1 = D_2g_2^{-1}, \text{ for any } g_1 \in D_1, g_2 \in D_2 \} \]

\[ m(D_1, D_2) := D_1D_2 \]

\[ \epsilon = \text{inclusion} \]

\[ D^{-1} := g^{-1}Dg^{-1} \]
Groupoid actions

$G \rightrightarrows X$ groupoid, $M$ set, $J : M \to X$ map.

\[ G \times_J M := \{ (g, m) \in G \times M \mid s(g) = J(m) \} \]

A (left) groupoid action of $G$ on $M$ with moment map $J : M \to X$ is a map $\Psi : G \times_J M \to M$, $\Psi(g, m) := g \cdot m$:

(i) $J(g \cdot m) = t(g),$

(ii) $gh \cdot m = g \cdot (h \cdot m),$

(iii) $(\epsilon(J(m))) \cdot m = m.$
Examples

1.) $G \rightarrow X$ acts on $G$ by left multiplication with moment map $t$.

2.) $G \rightarrow X$ acts on $X$ with moment map $\text{id}_X$, where $g \cdot s(g) := t(g)$. 
3.) **The $G$-action groupoid acts on $G$-spaces.** $G$ acts on two sets $M$ and $N$, $J : M \to N$ an equivariant map $J$ induces an action of the action groupoid $G \times N \rightrightarrows N$ on $M$:

$$(G \times N) \times_J M = \{((g,J(m)),m) \mid g \in G,m \in M\}$$

$$\subset (G \times N) \times M.$$ 

The action $\Psi : (G \times N) \times_J M \to M$ with moment $J$ is $J(((g,J(m)),m) := g \cdot m$. 

4.) **The Baer groupoid acts on $G$-spaces.** Let $G$ be a Lie group, $M$ a $G$-space, and $B : m \in M \mapsto G_m \in \mathcal{G}(G)$.

$$
\mathcal{B}(G) \times_B M := \{(gG_m, m) \in \mathcal{B}(G) \times M \mid m \in M\}
$$

$\mathcal{B}(G) \times_B M \to M$ given by $(gG_m, m) \mapsto g \cdot m$ defines an action of the Baer groupoid $\mathcal{B}(G) \rightrightarrows \mathcal{G}(G)$ on the $G$-space $M$ with moment map $B$. Notice that the level sets of the moment map are the isotropy type subsets $M_H$. 

A groupoid model for the optimal momentum map

\[(M, \omega, G, J), \ M \text{ connected}\]

\[T^*G \simeq G \times g^* \Rightarrow g^* \text{ action groupoid associated to the affine action of } G \text{ on } g^*\]

\[T^*G \times \mathcal{B}(G) \Rightarrow g^* \times S(G) \text{ product groupoid}\]

\[\Gamma \Rightarrow g^* \times S(G) \text{ wide subgroupoid defined by}\]

\[\Gamma := \{((g, \mu), gH) \mid g \in G, \mu \in g^*, H \in S(G)\}\]
\[ \Gamma \Rightarrow \mathfrak{g}^* \times \mathcal{S}(G) \text{ acts on } M \text{ with moment map } \mathcal{J} : M \rightarrow \mathfrak{g}^* \times \mathcal{S}(G), \mathcal{J}(m) = (\mathbf{J}(m), G_m) \]

\( \mathcal{J} \) has the Noether property and encodes both the conservation of the standard momentum and the law of conservation of the isotropy

Universality of \( \mathcal{J} \): \( \varphi(\mathcal{J}(m)) = \mathcal{J}(m) = (\mathbf{J}(m), G_m) \)

\[ M \xrightarrow{\mathcal{J}} \mathfrak{g}^* \times \mathcal{S}(G) \]

\[ M/A_G' \xrightarrow{\mathcal{J}} \]

\[ M_{G_m} \text{ and } \mathbf{J}^{-1}(\mu) \cap M_{G_m} \text{ connected } \Rightarrow \varphi \text{ injective.} \]
(\(M, \{\cdot, \cdot}\)\)) Poisson manifold

\(G\) acts canonically and properly on \(M\)

\(J : M \rightarrow M/A'_G\) optimal momentum map

For any \(\rho \in M/A'_G\) whose isotropy subgroup \(G_\rho\) acts properly on \(J^{-1}(\rho)\) we have:

\[(i) \quad M_\rho := J^{-1}(\rho)/G_\rho\] is a smooth symplectic regular quotient manifold:

\[(\pi^*_\rho \omega_\rho)(m)(X_f(m), X_h(m)) = \{f, h\}(m),\]
for any $f, h \in C^\infty(M)^G$, where $\pi_\rho : J^{-1}(\rho) \to M_\rho$ is the canonical projection. The pair $(M_\rho, \omega_\rho)$ is the optimal reduced space of $(M, \{\cdot, \cdot\})$ at $\rho$.

(ii) Let $h \in C^\infty(M)^G$. The flow $F_t$ of $X_h$ leaves $J^{-1}(\rho)$ invariant, commutes with the $G$-action, and therefore induces a flow $F_t^\rho$ on $M_\rho$ uniquely determined by the relation

$$\pi_\rho \circ F_t \circ i_\rho = F_t^\rho \circ \pi_\rho,$$

where $i_\rho : J^{-1}(\rho) \hookrightarrow M$ is the inclusion.
(iii) The flow $F_t^\rho$ in $(M_\rho, \omega_\rho)$ is Hamiltonian with the Hamiltonian function $h_\rho \in C^\infty(M_\rho)$ given by the equality $h_\rho \circ \pi_\rho = h \circ i_\rho$.

(iv) Let $k \in C^\infty(M)^G$ be another $G$-invariant function on $M$ and $\{\cdot, \cdot\}_{M_\rho}$ the Poisson bracket associated to the symplectic form $\omega_\rho$ on $M_\rho$. Then $\{h, k\}_\rho = \{h_\rho, k_\rho\}_{M_\rho}$.

If $G = \{e\}$ the distribution $A'_G$ coincides with the characteristic distribution of the Poisson manifold. The level sets of the optimal momentum map, and thereby the symplectic quotients $M_\rho$, are exactly the symplectic leaves of the Poisson manifold $(M, \{\cdot, \cdot\})$. 
\((M, \omega, G, J : M \to g^*)\).

If \(J(m) = \mu \in g^*, \mathcal{J}(m) = \rho \in M/A_G\)

\[
\mathcal{J}^{-1}(\rho) = \frac{(J^{-1}(\mu) \cap M_{G_m})^m}{NG_\mu(G_m)c(m)/G_m}
\]

- If action is free and proper, get usual reduced spaces.

- If freeness is dropped, get the strata of the singular reduced spaces.
• If $G$ is discrete, the optimal reduced spaces are (up to connected components) the symplectic manifolds $M_H/(N(H)/H)$.

• Properness of the $G_\rho$-action on $\mathcal{J}^{-1}(\rho)$ is a real hypothesis.
Optimal point Sjamaar principle

\[ \mathcal{J} : M \rightarrow M/A'_G, \, \rho \in M/A'_G \text{ a value of} \, \mathcal{J} \]

\[ H \subset G \text{ the unique } G\text{-isotropy subgroup such that } \mathcal{J}^{-1}(\rho) \subset M_H \text{ and } G_{\rho} \subset H \]

\[ N(H)\text{-action on } M_H \text{ induces a free action of } L := N(H)/H \text{ on } M_H \]

\[ M_H^{\rho} \text{ the unique connected component of } M_H \text{ that contains } \mathcal{J}^{-1}(\rho), \, L^{\rho} \text{ the closed subgroup of } L \text{ that leaves it} \]
invariant. Then

\[ L^\rho = N(H)^\rho / H \]

for some closed subgroup \( N(H)^\rho \) of \( N(H) \).

\( M^\rho_H \) is a symplectic embedded submanifold of \( M \) on which the group \( L^\rho \) acts freely and canonically. Let \( J_{L^\rho} : M^\rho_H \to M^\rho_H/A_{L^\rho}^l \) be the associated optimal momentum map.

\( i^\rho_H : M^\rho_H \hookrightarrow M \) inclusion
(i) \( T_{z\iota_H}^{\rho} (A'_{L\rho}(z)) = A'_{G}(z) \) for any \( z \in M_H^{\rho} \).

(ii) Let \( z \in J^{-1}(\rho) \) be such that \( J_{L\rho}(z) =: \sigma \in M_H^{\rho}/A'_{L\rho} \). Then \( J^{-1}(\rho) = J^{-1}_{L\rho}(\sigma) \).

(iii) \( L_\sigma^{\rho} = G_\rho/H \).
(iv) We have

\[(M_{H}^{\rho})_{\sigma} = \mathcal{J}_{L^{\rho}}^{-1}(\sigma)/L_{\sigma}^{\rho} = \mathcal{J}^{-1}(\rho)/(G_{\rho}/H)\]

\[= \mathcal{J}^{-1}(\rho)/G_{\rho} = M_{\rho}\]

If \(G_{\rho}\) acts properly on \(\mathcal{J}^{-1}(\rho)\) this equality is true symplectically, that is,

\[(M_{\rho}, \omega_{\rho}) = (((M_{H}^{\rho})_{\sigma}, (\omega|_{M_{H}^{\rho}})_{\sigma}).\]
Optimal orbit Poisson reduction

For $\rho \in M/A'_G$ let $O_\rho := G \cdot \rho \subset M/A'_G$

Assume that $G_\rho$ acts properly on $\mathcal{J}^{-1}(\rho)$

$\mathcal{J}^{-1}(O_\rho)$ has a unique smooth structure relative to which it is an initial submanifold of $M$. This structure coincides with the one that makes it diffeomorphic to $G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$. 
The restricted $G$-action on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is smooth and proper and all its isotropy subgroups are conjugate to a given compact isotropy subgroup of the $G$-action on $M$.

The **optimal orbit reduced space** $M_{\mathcal{O}_\rho} := \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ admits a unique symplectic structure $\omega_{\mathcal{O}_\rho}$ that makes it symplectomorphic to the point reduced space $M_\rho$:

$$\mathcal{J}^{-1}(\mathcal{O}_\rho)/G \simeq G \times_{G_\rho} \mathcal{J}^{-1}(\rho)/G \simeq \mathcal{J}^{-1}(\rho)/G_\rho$$
Reduced dynamics

(i) Let $h \in C^\infty(M)^G$. The flow $F_t$ of $X_h$ leaves $J^{-1}(O_\rho)$ invariant, commutes with the $G$-action, and therefore induces a flow $F_t^{O_\rho}$ on $M_{O_\rho}$ uniquely determined by the relation

$$\pi_{O_\rho} \circ F_t \circ i_{O_\rho} = F_t^{O_\rho} \circ \pi_{O_\rho},$$

where $i_{O_\rho} : J^{-1}(O_\rho) \hookrightarrow M$ is the inclusion.

(ii) The flow $F_t^{O_\rho}$ on $(M_{O_\rho}, \omega_{O_\rho})$ is Hamiltonian with the Hamiltonian function $h_{O_\rho} \in C^\infty(M_{O_\rho})$ given by the equality $h_{O_\rho} \circ \pi_{O_\rho} = h \circ i_{O_\rho}$. 
(ii) Let $k \in C^\infty(M)^G$ be another $G$-invariant function on $M$ and $\{\cdot, \cdot\}_{\mathcal{O}_\rho}$ the Poisson bracket associated to the symplectic form $\omega_{\mathcal{O}_\rho}$ on $M_{\mathcal{O}_\rho}$. Then, $\{h, k\}_{\mathcal{O}_\rho} = \{h_{\mathcal{O}_\rho}, k_{\mathcal{O}_\rho}\}_{\mathcal{O}_\rho}$. 
Optimal orbit Sjamaar principle

• $H \subset G$ the unique $G$-isotropy subgroup such that $J^{-1}(\rho) \subset M_H$ and $G_\rho \subset H$

• $G_\rho$ acts properly on $J^{-1}(\rho)$

• $M^\rho_H$ connected component of $M_H$ that contains $J^{-1}(\rho)$

• $N(H)^{\rho}$ the closed subgroup of the normalizer $N(H)$ of $H$ that leaves $M^\rho_H$ invariant

• $L^\rho := N(H)^{\rho}/H$
(i) Let $z \in J^{-1}(\rho)$ be such that $J_{L\rho}(z) =: \sigma \in M_H^\rho/A'_L^\rho$ and $N_\rho := N(H)^\rho \cdot \rho \subset M/A'_G$. The set $J_{L\rho}^{-1}(L^\rho \cdot \sigma) = J^{-1}(N_\rho)$ is an embedded submanifold of $J^{-1}(O_\rho)$.

(ii) $J^{-1}(O_\rho) = \coprod_{[g] \in G/N(H)^\rho} J^{-1}(N_g \cdot \rho)$

(iii) $(J_{L\rho}^{-1}(L^\rho \cdot \sigma)/L^\rho, (\omega|_{M_H^\rho})_{L^\rho \cdot \sigma}) 
\cong (J^{-1}(O_\rho)/G, \omega_{O_\rho})$
Polar reduction

\( \mathcal{J} : M \to M/A'_G, \ \rho \in M/A'_G, \ G_\rho \) closed in \( G \)

- \( A'_G \) restricts to a smooth integrable regular distribution on \( \mathcal{J}^{-1}(\mathcal{O}_\rho) \), also denoted \( A'_G \)
- \( M'_{\mathcal{O}_\rho} := \mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G \) admits a unique manifold quotient structure; \( M'_{\mathcal{O}_\rho} \) is diffeomorphic to \( G/G_\rho \)
- Projection \( \mathcal{J}_{\mathcal{O}_\rho} : \mathcal{J}^{-1}(\mathcal{O}_\rho) \to \mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G \) is a smooth surjective submersion.
(M, ω) symplectic. There is a unique presymplectic form \( \omega'_\mathcal{O}_\rho \) on \( M'_\mathcal{O}_\rho \cong G/G_\rho \) that satisfies

\[
i^*_\mathcal{O}_\rho \omega = \pi^*_\mathcal{O}_\rho \omega_{\mathcal{O}_\rho} + \mathcal{I}^*_\mathcal{O}_\rho \omega'_{\mathcal{O}_\rho}.
\]

\( \omega'_{\mathcal{O}_\rho} \) symplectic \( \iff \exists z \in \mathcal{J}^{-1}(\mathcal{O}_\rho) \) (so for all)

\[
g \cdot z \cap (g \cdot z)^\omega \subset T_z M_{G_z}^z
\]

If \( J : M \rightarrow g^* \), \( M \) connected, \( J(z) = \mu \in g^* \) and \( G_z = H \).

Condition: \( g_\mu = \text{Lie}(N_{G_\mu}(H)) \).
**SINGULAR POINT REDUCTION**

\[ J : (M, \omega) \to g^*, \]

\[ \sigma(g) := J(g \cdot m) - Ad_{g^{-1}}^* J(m), \quad \sigma : G \to g^*, \]

\[ \Theta(g, \nu) := Ad_{g^{-1}}^* \nu + \sigma(g), \]

\[ H := G_z, \]

\[ M_H^z \text{ connected component of } M_H \text{ containing } z, \quad J(z) = \mu, \]

\[ G_\mu \text{ is the } \Theta\text{-isotropy at } \mu. \]
Singular symplectic point strata

(i) \( J^{-1}(\mu) \cap (G_\mu \cdot M^\sim_H) \) embedded in \( M \)

(ii) \( M^{(H)}_\mu := [J^{-1}(\mu) \cap (G_\mu \cdot M^\sim_H)]/G_\mu \) has a unique quotient manifold structure such that

\[
\pi^{(H)}_\mu : J^{-1}(\mu) \cap (G_\mu \cdot M^\sim_H) \longrightarrow M^{(H)}_\mu
\]

is a surjective submersion
(iii) Unique symplectic structure $\omega^{(H)}_\mu$ on $M^{(H)}_\mu$

$$\iota^{(H)}_\mu \ast \omega = \pi^{(H)}_\mu \ast \omega^{(H)}_\mu,$$

$\iota^{(H)}_\mu : J^{-1}(\mu) \cap (G_\mu \cdot M_H^z) \hookrightarrow M$ inclusion. $(M^{(H)}_\mu, \omega^{(H)}_\mu)$ are the singular symplectic point strata.

(iv) $h \in C^\infty(M)^G$. The flow $F_t$ of $X_h$ leaves the connected components of $J^{-1}(\mu) \cap (G_\mu \cdot M_H^z)$ invariant and commutes with the $G_\mu$-action, so it induces a flow $F^\mu_t$ on $M^{(H)}_\mu$

$$\pi^{(H)}_\mu \circ F_t \circ \iota^{(H)}_\mu = F^\mu_t \circ \pi^{(H)}_\mu.$$
(v) $F^H_t$ is Hamiltonian on $M^{(H)}_\mu$, with reduced Hamiltonian function $h^{(H)}_\mu : M^{(H)}_\mu \to \mathbb{R}$

$$h^{(H)}_\mu \circ \pi^{(H)}_\mu = h \circ i^{(H)}_\mu.$$ 

$X_h$ and $X_{h^{(H)}_\mu}$ are $\pi^{(H)}_\mu$-related.

(vi) $h, k \in C^\infty(M)^G \Rightarrow \{h, k\} \in C^\infty(M)^G$

$$\{h, k\}^{(H)}_\mu = \{h^{(H)}_\mu, k^{(H)}_\mu\}^{(H)}_{M^{(H)}_\mu}$$

$\{\cdot, \cdot\}^{(H)}_{M^{(H)}_\mu}$ is the Poisson bracket induced by the symplectic structure on $M^{(H)}_\mu$. 

Sjamaar point reduction principle

• Remember that \( H := G_z \)

• \( N(H)^z \subset N(H) \) all elements that leave \( M^z_H \) invariant. \( N(H)^z \) is open hence closed in \( N(H) \). Also \( H \subset N(H)^z \). Thus \( \text{Lie}(N(H)^z/H) = \text{Lie}(N(H)/H) =: l \)

• \( L^z := N(H)^z/H \) acts freely and canonically on \( M^z_H \) with momentum map

\[
J_{L^z} : z' \in M^z_H \mapsto \Lambda(J|_{M^z_H}(z') - \mu) \in (\text{Lie}(L^z))^*
\]

where \( \mu := J(z) \in g^* \).
\[
\Lambda : (g_z^0)^H \rightarrow (\text{Lie}(L^z))^* \text{ denotes the natural } L^z\text{-equivariant isomorphism given by }
\]

\[
\left\langle \Lambda(\beta), \left. \frac{d}{dt} \right|_{t=0} (\exp t \xi)H \right\rangle = \langle \beta, \xi \rangle,
\]

\(\beta \in (g_z^0)^H, \, \xi \in \text{Lie}(N(H)^z) = \text{Lie}(N(H))\)

- \(g_z^0\) denotes the annihilator of \(g_z\) in \(g^*\)
- \((g_z^0)^H\) are the \(H\)-fixed points in \(g_z^0\)
- Non-equivariance one-cocycle of \(J_{L^z}\)

\[
\tau : l \in L^z \mapsto \Lambda(\sigma(n) + n \cdot \mu - \mu) \in (\text{Lie}(L^z))^*
\]

for \(l = nH \in L^z\) and \(n \in N(H)^z\).
(i) \( \pi^{(H)}_{\mu}: J^{-1}(\mu) \cap (G_\mu \cdot M^\tilde{\mu}_H) \to M^{(H)}_{\mu} := [J^{-1}(\mu) \cap (G_\mu \cdot M^\tilde{\mu}_H)]/G_\mu \) is a smooth fiber bundle with fiber \( G_\mu/H \) and structure group \( N_{G_\mu}(H)^\tilde{\mu}/H \).

(ii) \( (M^\tilde{\mu}_H)_0 := J^{-1}_{L^\tilde{\mu}}(0)/L^\tilde{\mu}_0 \)

\[ = [J^{-1}(\mu) \cap M^\tilde{\mu}_H]/(N_{G_\mu}(H)^\tilde{\mu}/H) \]

\( L^\tilde{\mu}_0 \neq L^\tilde{\mu} \), in general; the action is affine.
(iii) $\pi_0 : J_{L_0}^{-1}(0) \to (M_H^\tilde{z})_0$ is a principal $L_0$-bundle. $G_\mu/H$ is a right $(N_{G_\mu}(H)^\tilde{z}/H)$-space and $J^{-1}(\mu) \cap M_H^\tilde{z}$ is a left $(N_{G_\mu}(H)^\tilde{z}/H)$-space. Associated bundle with fiber $G_\mu/H$

$$G_\mu/H \times_{N_{G_\mu}(H)^\tilde{z}/H} (J^{-1}(\mu) \cap M_H^\tilde{z}) \rightarrow [J^{-1}(\mu) \cap M_H^\tilde{z}]/(N_{G_\mu}(H)^\tilde{z}/H).$$

is $G_\mu$-symplectomorphic to

$$\pi^{(H)}_\mu : J^{-1}(\mu) \cap (G_\mu \cdot M_H^\tilde{z}) \rightarrow M_\mu^{(H)},$$

that is,
\[ G_\mu / H \times _{N_{G_\mu} (H)} (H)^z / H (J^{-1}(\mu) \cap M_H^{\tilde{z}}) \] is \( G_\mu \)-diffeomorphic to \( J^{-1}(\mu) \cap (G_\mu \cdot M_H^{\tilde{z}}) \)

\[ (M_H^{\tilde{z}})_0 = J_{L^{\tilde{z}}}^{-1}(0) / L_{0^{\tilde{z}}} = \]

\[ (J^{-1}(\mu) \cap M_H^{\tilde{z}}) / (N_{G_\mu} (H)^z / H) \]

is symplectomorphic to \( M_\mu^{(H)} \).

Up to connected components, point strata are symplectomorphic to the corresponding optimal reduced spaces. So, optimal reduction, which is always regular, directly yields the strata of the singular reduced spaces.
• \( \{ J^{-1}(\mu) \cap (G_{\mu} \cdot M^z_H) \mid J(z) = \mu \} \) forms a Whitney (B) stratification of \( J^{-1}(\mu) \).

• \( \{ M_{\mu}^{(H)} \mid (H) \} \) is a symplectic Whitney (B) stratification of the cone space \( M_{\mu} := J^{-1}(\mu)/G_{\mu} \).

• Each connected component of \( M_{\mu} \) contains a unique open stratum that is connected, open, and dense in the connected component of \( M_{\mu} \) that contains it.
SINGULAR ORBIT REDUCTION

\((M, \omega, G, \mathbf{J} : M \to \mathfrak{g}^*)\), affine action \(\Theta\) associated to group one-cocycle \(\sigma : G \to \mathfrak{g}^*\), induced \(\Sigma \in Z^2(\mathfrak{g}; \mathbb{R})\)

\[
\Sigma(\xi, \eta) := \mathbf{J}^{[\xi, \eta]}(z) - \{\mathbf{J}^\xi, \mathbf{J}^\eta\}(z), \quad z \in M
\]

Affine Lie-Poisson structure on \(\mathfrak{g}^*\)

\[
\{f, g\}_\Sigma^\pm(\mu) := \pm \left\langle \mu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle \pm \Sigma \left( \frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right)
\]
Orbit symplectic form

\[ \omega_{\mathcal{O}_\mu}^{\pm}(\nu)(\xi_{\mathfrak{g}}^*(\nu), \eta_{\mathfrak{g}}^*(\nu)) = \pm \langle \nu, [\xi, \eta] \rangle \mp \Sigma(\xi, \eta), \]

where \( \xi_{\mathfrak{g}}^*(\nu) := -\text{ad}^*_\xi \nu + \Sigma(\xi, \cdot). \)

If \( G \)-action is free proper and \( \mathcal{O}_\mu \) is an embedded sub-manifold of \( \mathfrak{g}^* \) then \( M_{\mathcal{O}_\mu} := J^{-1}(\mathcal{O}_\mu)/G \) is a symplectic manifold,

\[ i_{\mathcal{O}_\mu}^* \omega = \pi_{\mathcal{O}_\mu}^* \omega_{\mathcal{O}_\mu} + J_{\mathcal{O}_\mu}^* \omega_{\mathcal{O}_\mu}^+, \]

\( J_{\mathcal{O}_\mu} = J|_{J^{-1}(\mathcal{O}_\mu)}, \ i_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \hookrightarrow M, \ \pi_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \to M_{\mathcal{O}_\mu}. \)
Singular case: what topology on $J^{-1}(\mathcal{O}_\mu)$?

For $J^{-1}(\mu)$, induced topology. Wrong now!

Initial topology induced by the map $J_{\mathcal{O}_\mu} := J|_{J^{-1}(\mathcal{O}_\mu)} : J^{-1}(\mathcal{O}_\mu) \to \mathcal{O}_\mu$, where $\mathcal{O}_\mu$ comes with its own structure diffeomorphic to $G/G_{\mu}$

$$f : [g, z] \in G \times G_{\mu} \mapsto g \cdot z \in J^{-1}(\mathcal{O}_\mu)$$

homeomorphism. Consistent with regular case.
(i) \( G \cdot (J^{-1}(\mu) \cap M_H^\tilde{z}) \) is an initial submanifold of \( M \)

\[
T_m (G \cdot (J^{-1}(\mu) \cap M_H^\tilde{z})) = \text{span} \{ \xi_M(m) + X_f(m) \mid \xi \in \mathfrak{g}, f \in C^\infty(M)^G \} = \mathfrak{g} \cdot m + A_G'(m),
\]

\( A'_G \) the polar distribution of \( G \)-action on \( M \).

(ii) \( M_{O_\mu}^{(H)} := [G \cdot (J^{-1}(\mu) \cap M_H^\tilde{z})]/G \) has a unique quotient differentiable structure such that the projection

\[
\pi_{O_\mu}^{(H)} : G \cdot (J^{-1}(\mu) \cap M_H^\tilde{z}) \longrightarrow M_{O_\mu}^{(H)}
\]

is a surjective submersion.
(iii) There is a unique symplectic structure $\omega^{(H)}_{O_\mu}$ on $M^{(H)}_{O_\mu}$ characterized by

$$i^{(H)}_{O_\mu} \ast \omega = \pi^{(H)}_{O_\mu} \ast \omega^{(H)}_{O_\mu} + J^{(H)}_{O_\mu} \ast \omega^+_{O_\mu},$$

where $i^{(H)}_{O_\mu} : G \cdot (J^{-1}(\mu) \cap M^\nu_H) \hookrightarrow M$ is the inclusion and $J^{(H)}_{O_\mu} : G \cdot (J^{-1}(\mu) \cap M^\nu_H) \rightarrow O_\mu$ is obtained by restriction of the momentum map $J$.

(iv) Let $h \in C^\infty(M)^G$. The flow $F_t$ of $X_h$ leaves the connected components of $G \cdot (J^{-1}(\mu) \cap M^\nu_H)$ invariant and commutes with the $G$-action, so it induces a flow $F_t^{O_\mu}$ on $M^{(H)}_{O_\mu}$

$$\pi^{(H)}_{O_\mu} \circ F_t \circ i^{(H)}_{O_\mu} = F_t^{O_\mu} \circ \pi^{(H)}_{O_\mu}.$$
(v) The flow $F_{t}^{O_{\mu}}$ is Hamiltonian on $(M^{(H)}_{O_{\mu}}, \omega^{(H)}_{O_{\mu}})$ relative to the reduced Hamiltonian $h_{O_{\mu}}^{(H)} : M^{(H)}_{O_{\mu}} \to \mathbb{R}$ defined by

$$h_{O_{\mu}}^{(H)} \circ \pi^{(H)}_{O_{\mu}} = h \circ i_{O_{\mu}}^{(H)}.$$ 

$X_{h}$ and $X_{h_{O_{\mu}}^{(H)}}$ are $\pi_{O_{\mu}}^{(H)}$-related.

(vi) $h, k \in C^{\infty}(M)^{G} \Rightarrow \{h, k\} \in C^{\infty}(M)^{G}$

$$\{h, k\}_{O_{\mu}}^{(H)} = \{h_{O_{\mu}}^{(H)}, k_{O_{\mu}}^{(H)}\}_{M^{(H)}_{O_{\mu}}},$$

where $\{\cdot, \cdot\}_{M^{(H)}_{O_{\mu}}}$ denotes the Poisson bracket induced by the symplectic structure in $M^{(H)}_{O_{\mu}}$. 
Sjamaar orbit reduction principle

(i) \[ \pi_{O_{\mu}}^{(H)} : G \cdot (J^{-1}(\mu) \cap M^Z_H) \longrightarrow \]

\[ M_{O_{\mu}}^{(H)} = [G \cdot (J^{-1}(\mu) \cap M^Z_H)]/G \]

defines a smooth fiber bundle with fiber \(G/H\) and structure group \(N(H)^Z/H\). Recall that \(N(H)^Z\) is the open and hence closed subgroup of \(N(H)\) that leaves \(M^Z_H\) invariant (as a set) and \(J(z) = \mu\).
(ii) \( L^z := N(H)^z / H \) acts freely properly and canonically on \( M_H^z \). It admits an associated momentum map

\[
J_{L^z} : m \in M_H^z \mapsto \Lambda(J_{|M_H^z}(m) - \mu) \in \mathfrak{l}^*.
\]

The regular orbit reduced space \((M_H^z)_{O_0}\) at the affine orbit corresponding to \( 0 \in \mathfrak{l}^* \) is given by

\[
(M_H^z)_{O_0} = J_{L^z}^{-1}(O_0)/L^z = [J^{-1}(N(H)^z \cdot \mu) \cap M_H^z] / (N(H)^z / H)
\]
(iii) \( \pi_{\mathcal{O}_0} : J^{-1}_{L^\mathbb{C}}(\mathcal{O}_0) \to (M^\mathbb{C}_H)_{\mathcal{O}_0} \) is a principal \( L^\mathbb{C} \)-bundle. \( G/H \) is a right \( (N(H)^\mathbb{C}/H) \)-space and \( J^{-1}(N(H)^\mathbb{C} \cdot \mu) \cap M^\mathbb{C}_H \) is a left \( (N(H)^\mathbb{C}/H) \)-space. The associated bundle

\[
G/H \times_{N(H)^\mathbb{C}/H} (J^{-1}(N(H)^\mathbb{C} \cdot \mu) \cap M^\mathbb{C}_H) \\
\rightarrow [J^{-1}(N(H)^\mathbb{C} \cdot \mu) \cap M^\mathbb{C}_H] / (N(H)^\mathbb{C}/H).
\]

is \( G \)-symplectomorphic to

\[
\pi_{\mathcal{O}_\mu}^{(H)} : G \cdot (J^{-1}(\mu) \cap M^\mathbb{C}_H) \rightarrow M^{(H)}_{\mathcal{O}_\mu},
\]
that is,

- $G/H \times_{N(H)} (J^{-1}(N(H) \cdot \mu) \cap M_{H}^{\tilde{z}})$ is $G$-diffeomorphic to $G \cdot (J^{-1}(\mu) \cap M_{H}^{\tilde{z}})$

- $(M_{H}^{\tilde{z}})_{0} = J^{-1}_{L^{\tilde{z}}}(O_{0})/L^{\tilde{z}} = \left[ J^{-1}(N(H) \cdot \mu) \cap M_{H}^{\tilde{z}} \right] / (N(H) / H)$

is symplectomorphically to $M_{O_{\mu}}^{(H)}$. 
Let $l_{\mu} : J^{-1}(\mu) \hookrightarrow J^{-1}(O_\mu)$ be the inclusion and $L_{\mu} : J^{-1}(\mu)/G_{\mu} \rightarrow J^{-1}(O_\mu)/G$ the map defined by the commutative diagram

\[
\begin{array}{ccc}
J^{-1}(\mu) & \xrightarrow{l_{\mu}} & J^{-1}(O_\mu) \\
\pi_{\mu} \downarrow & & \downarrow \pi_{O_\mu} \\
J^{-1}(\mu)/G_{\mu} & \xrightarrow{L_{\mu}} & J^{-1}(O_\mu)/G.
\end{array}
\]

Consider $J^{-1}(\mu)/G_{\mu}$ as a smooth symplectically stratified topological space. Then:
(i) \( \{ \mathcal{M}_{\mathcal{O}_\mu}^{(H)} | (H) \} \) is a smooth symplectic Whitney (B) stratification on \( J^{-1}(\mathcal{O}_\mu)/G \).

(ii) \( L_\mu \) is a homeomorphism of smooth symplectic Whitney (B) stratified spaces.
\[ l_\mu : J^{-1}(\mu) \hookrightarrow J^{-1}(O_\mu) \text{ is the inclusion} \]

\[ L_\mu \text{ is an isomorphism of symplectic cone (hence Whitney (B)) stratified spaces; in particular it is a homeomorphism} \]

\[ L_0 : J_{L^Z}(0)/L_0^Z \rightarrow J_{L^Z}(O_0)/L^Z \text{ and} \]
\[ L^{(H)}_\mu : J^{-1}(\mu) \cap (G_\mu \cdot M^Z_H)/G_\mu \rightarrow G \cdot (J^{-1}(\mu) \cap M^Z_H)/G \text{ are symplectomorphisms} \]

\[ L^{(H)}_\mu \text{ is the restriction of } L_\mu \text{ to the stratum determined by } H := G_m \]
\( \pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu \) and
\( \pi_{O_\mu} : J^{-1}(O_\mu) \to J^{-1}(O_\mu)/G \)
are projections onto quotient manifolds

\[
\begin{align*}
  f^{(H)}_{\mu} & : J_{L_z}(0)/L_0^z \to J^{-1}(\mu) \cap (G_\mu \cdot M_{H_\mu}^z)/G_\mu \\
  f^{(H)}_{O_\mu} & : J_{L_z}(O_0)/L^z \to G \cdot (J^{-1}(\mu) \cap M_{H}^z)/G 
\end{align*}
\]
are the Sjamaar principle diffeomorphisms

Second pair of upwards pointing arrows are the inclusions of the stratum in the ambient stratified space.
POISSON REDUCTION

$(M, \{\cdot, \cdot\})$ Poisson, $G$ acts canonically freely properly on $M$; $M/G$ smooth manifold; $\pi : M \to M/G$ smooth surjective submersion.

$\mathcal{J} : M \to M/A'_G$ optimal momentum

(i) $M/G$ Poisson manifold

$$\{f, g\}^{M/G}(\pi(m)) = \{f \circ \pi, g \circ \pi\}(m),$$

for any $m \in M$ and $f, g \in C^\infty(M/G)$. 
(ii) $\{\cdot, \cdot\}^{M/G}$ is the only Poisson structure for which $\pi: (M, \{\cdot, \cdot\}) \to (M/G, \{\cdot, \cdot\}^{M/G})$ is a Poisson map.

(iii) $h \in C^\infty(M)^G$. The Hamiltonian flow $F_t$ of $X_h$ commutes with the $G$-action, so it induces a flow $F_t^{M/G}$ on $M/G$ by

$$\pi \circ F_t = F_t^{M/G} \circ \pi.$$ 

$F_t^{M/G}$ is Hamiltonian on $(M/G, \{\cdot, \cdot\}^{M/G})$ for $[h] \in C^\infty(M/G)$ defined by

$$[h] \circ \pi = h.$$

The vector fields $X_h$ and $X_{[h]}$ are $\pi$-related.
(iv) The symplectic leaves of \((M/G, \{\cdot, \cdot\}^{M/G})\) are given by the optimal orbit reduced spaces \((\mathcal{J}^{-1}(O_\rho)/G, \omega_{O_\rho}), \rho \in M/A'_G\).

(v) If \((M, \{\cdot, \cdot\})\) is symplectic with form \(\omega\) and the \(G\)-action has a standard momentum map \(J : M \to g^*\), then the symplectic leaves of \((M/G, \{\cdot, \cdot\}^{M/G})\) are given by the spaces \(\left(M^c_{O_\mu} := G \cdot J^{-1}(\mu)^c/G, \omega^c_{O_\mu}\right)\), where \(J^{-1}(\mu)^c\) is a connected component of the fiber \(J^{-1}(\mu)\) and \(\omega^c_{O_\mu}\) the restriction to \(M^c_{O_\mu}\) of the symplectic form \(\omega_{O_\mu}\) of the orbit reduced space \(M_{O_\mu}\). If \(G\) is compact, \(M\) is connected, and \(J\) is proper, then \(M^c_{O_\mu} = M_{O_\mu}\).
Distributional Poisson reduction

$M$ manifold, $S \subset M$ with its own topology.

$S$ is **decomposed** with pieces $\{S_i\}_{i \in I}$ if:

- $\{S_i\}_{i \in I}$ locally finite partition of $S$
- each $S_i$ is a manifold with its own topology
- each $S_i$ is locally closed in $S$
• Frontier condition: \( S_i \cap \overline{S}_j \neq \emptyset \implies S_i \subset \overline{S}_j \)

\[ D \subset TM|_S \text{ is a smooth distribution on } S \text{ adapted to the decomposition } \{S_i\}_{i \in I}, \text{ if } D \cap TS_i \text{ is a smooth distribution on each } S_i. \]

\( D \) is integrable if each \( D \cap TS_i \) is integrable.
Integrability of $D_{S_i} := D \cap T S_i$ on $S_i$ partitions each $S_i$ into its maximal integral manifolds. Get equivalence relation $D_S$ on $S$ whose equivalence classes are these maximal integral manifolds:

$$\pi_{D_S} : S \to S/D_S := \bigcup_{i \in I} S_i/D_{S_i}$$
\((M, \{\cdot, \cdot\})\) Poisson. \(D \subset TM\) smooth distribution is \textbf{Poisson} or \textbf{canonical}, if \(df|_D = dg|_D = 0\), for any \(f, g \in C^\infty_M(U)\) and any open subset \(U \subset P \implies d\{f, g\}|_D = 0\).

If \(D\) is spanned by a family of infinitesimal Poisson automorphisms then \(D\) is a Poisson distribution. Converse is not necessarily true.

For \(V \subset S/D_S\) open define \(C^\infty_{S/D_S}(V)\) by:

\[ f \in C^\infty_{S/D_S}(V) \iff \forall z \in V, \exists m \in \pi^{-1}_{D_S}(V), \exists U_m \subset M\] open neighborhood of \(m\), \(\exists F \in C^\infty_M(U_m)\) such that

\[ f \circ \pi_{D_S}|_{\pi^{-1}_{D_S}(V) \cap U_m} = F|_{\pi^{-1}_{D_S}(V) \cap U_m}. \]
F local extension of $f \circ \pi_{DS}$ at $m \in \pi_{DS}^{-1}(V)$

$C_{S/D_S}^\infty$ has the $(D, D_S)$-local extension property if the topology of $S$ is stronger than the relative topology and the local extensions of $f \circ \pi_{DS}$ can be chosen to satisfy

$$dF(n)|_{D(n)} = 0, \quad \text{for any} \quad n \in \pi_{DS}^{-1}(V) \cap U_m.$$ 

$F$ is a local $D$-invariant extension of $f \circ \pi_{DS}$ at the point $m \in \pi_{DS}^{-1}(V)$. 
If $D_S$ is a smooth, integrable, and regular distribution on $S$, then the presheaf $\mathcal{C}^\infty_{S/D_S}$ coincides with the presheaf of smooth functions on $S/D_S$ when considered as a regular quotient manifold.

$(M, \{\cdot, \cdot\})$ Poisson, $S \subset M$ decomposed subset, $D \subset TM|_S$ a Poisson integrable generalized distribution adapted to the decomposition of $S$. Assume $\mathcal{C}^\infty_{S/D_S}$ has the $(D, D_S)$-local extension property.
The quadruple \((M, \{\cdot, \cdot\}, D, S)\) is **Poisson reducible** if 
\((S/D_S, C_S^{\infty}, \{\cdot, \cdot\}^{S/D_S})\) is a well defined presheaf of Poisson algebras where, for any open set \(V \subset S/D_S\), the bracket

\[
\{\cdot, \cdot\}^{S/D_S}_V : C_S^{\infty}(V) \times C_S^{\infty}(V) \to C_S^{\infty}(V)
\]

is given by

\[
\{f, g\}^{S/D_S}_V(\pi_D S(m)) := \{F, G\}(m),
\]

for any \(m \in \pi_D^{-1}(V)\), where \(F, G\) are local \(D\)-invariant extensions at \(m\) of \(f \circ \pi_D S\) and \(g \circ \pi_D S\), respectively.
If \( B^\#(\Delta_m) \subset [\Delta_m]^\circ \) for any \( m \in S \) then \( (M, \{\cdot, \cdot\}, D, S) \) is Poisson reducible. Here

\[
\Delta_m := \{ dF(m) \mid F \in C^\infty_M(U_m), dF(z)|_{D(z)} = 0, \text{ for all } z \in U_m \cap S, \text{ and for any open neighborhood } U_m \text{ of } m \text{ in } M \}
\]

\[
\Delta_m^S := \{ dF(m) \in \Delta_m \mid F|_{U_m \cap V_m} \text{ is constant for an open neighborhood } U_m \text{ of } m \text{ in } M \text{ and an open neighborhood } V_m \text{ of } m \text{ in } S \}.
\]

Condition is only sufficient.
Assume: $S \subset M$ embedded, $D \subset TM|_S$ a canonical sub-bundle such that $D_S := D \cap TS$ is a smooth, integrable, regular distribution on $S$.

Then $(M, \{\cdot, \cdot\}, D, S)$ is Poisson reducible $\iff$ 

$$B^\#(D^\circ) \subset TS + D.$$ 

**WARNING!** The class of functions used in Poisson reduction matters very much.

• $(M, \omega, D, J^{-1}(\mu))$ is Poisson reducible but

• $(M, \omega, D_{J^{-1}(\mu)}, J^{-1}(\mu))$ is **not**, even though the corresponding quotient manifolds are the same.
$S \subset M$ embedded and $D := B^\#((TS)^\circ) \subset TM|_S$. Assume that the characteristic distribution $D_S := D \cap TS$ is smooth, integrable such that $C^\infty_{S/D_S}$ has the $(D, D_S)$-local extension property. Then $(M, \{\cdot, \cdot\}, D, S)$ is Poisson reducible.
COISOYTROPIC REDUCTION

\[ S \subset M \text{ is } \text{coisotropic} \text{ if } B^\#((TS)^\circ) \subset TS. \]

If \( M \) is symplectic, then \( B^\#((TS)^\circ) = (TS)^\omega \) and this condition becomes \( (TS)^\omega \subset TS \)

coisotropic submanifolds \( \equiv \) first class constraints

The following are equivalent:
(i) $S$ is coisotropic;

(ii) if $f|_S \equiv 0$ then $X_f|_S \in \mathcal{X}(S)$;

(iii) for any $s \in S$, any open neighborhood $U_s$ of $s$ in $M$, and any $g \in C^\infty(U_s)$ such that $X_g(s) \in T_sS$, if $f \in C^\infty(U_s)$ satisfies $\{f, g\}(s) = 0$, then $X_f(s) \in T_sS$;

(iv) the subalgebra $\{f \in C^\infty(M) \mid f|_S \equiv 0\}$ is a Poisson subalgebra of $(C^\infty(M), \{\cdot, \cdot\})$. 
Let $S$ be an embedded coisotropic submanifold of $M$ and $D := B^\#((TS)^\circ)$. Then:

(i) $D = D \cap TS = D_S$ is a smooth generalized distribution on $S$.

(ii) $D$ is integrable.

(iii) If $C^\infty_{S/D_S}$ has the $(D, D_S)$-local extension property then $(M, \{\cdot, \cdot\}, D, S)$ is Poisson reducible.
$S \subset M$ embedded submanifold such that the characteristic distribution $D_S := B^{\#}((TS)^{\circ}) \cap TS$ is smooth, integrable, Poisson, regular on $S$. Even though the quotient manifolds associated to $(M, \{\cdot, \cdot\}, D, S)$ and $(M, \{\cdot, \cdot\}, D_S, S)$ are the same and $(M, \{\cdot, \cdot\}, D, S)$ is reducible (by the general proposition above), $(M, \{\cdot, \cdot\}, D_S, S)$ is, in general, not reducible. Its reducibility is equivalent to $S$ being a coisotropic submanifold of $M$. 


$S \subset (M, \{\cdot, \cdot\})$ embedded is **cosymplectic** if

**(i)** $B^\#((TS)^\circ) \cap TS = \{0\}$.

**(ii)** $T_S S + T_S \mathcal{L}_s = T_S M$,

for any $s \in S$ and $\mathcal{L}_s$ the symplectic leaf of $(M, \{\cdot, \cdot\})$ containing $s \in S$. 

**COSYMPELECTIC REDUCTION**
If \((M, \omega)\) is symplectic, then \(B^\#((TS)^\circ) = (TS)^\omega\) and \(\mathcal{L}_s\) is the connected component of \(M\) containing \(s\). Therefore the first condition says that \((TS)^\omega \cap TS = \{0\}\) and the second condition is vacuous.

Therefore \(S\) is cosymplectic if and only if \(S\) is a symplectic submanifold of \((M, \omega)\).

cosymplectic manifolds = second class constraints
\( S \subset M \) cosymplectic. Then for any \( s \in S \)

(i) \( T_s \mathcal{L}_s = (T_s S \cap T_s \mathcal{L}_s) \oplus B^\#(s)((T_s S)^\circ) \), where \( \mathcal{L}_s \) is the symplectic leaf containing \( s \in S \).

(ii) \( (T_s S)^\circ \cap \ker B^\#(s) = \{0\} \).

(iii) \( T_s M = B^\#(s)((T_s S)^\circ) \oplus T_s S \).

(iv) \( B^\#((TS)^\circ) \) is a subbundle of \( TM|_S \) and hence \( TM|_S = B^\#((TS)^\circ) \oplus TS \).
(v) The symplectic leaves intersect \( S \) transversely and hence \( S \cap L \) is an initial submanifold of \( S \), for any symplectic leaf \( L \) of \((M, \{\cdot, \cdot\})\).
Weinstein: \( S \) a cosymplectic submanifold of \( M \). Let \( D := B^\#((TS)^\circ) \subset TM|_S \). Then

(i) \((M, \{\cdot, \cdot\}, D, S)\) is Poisson reducible.

(ii) The quotient manifold equals \( S \) and the reduced bracket \( \{\cdot, \cdot\}^S \) is given by

\[
\{f, g\}^S(s) = \{F, G\}(s),
\]

\(f, g \in C^\infty_{S,M}(V)\) and \(F, G \in C^\infty_M(U)\) are local \( D \)-invariant extensions around \( s \in S \).
(iii) If \( f \in C_{S,M}^\infty(V) \) then \( X_f \) is given by

\[
Ti \circ X_f = X_F \circ i,
\]

where \( F \in C_M^\infty(U) \) is a local \( D \)-invariant extension of \( f \) and \( i : S \hookrightarrow M \) is the inclusion.

(iv) If \( f \in C_{S,M}^\infty(V) \) then \( X_f \) is given by

\[
Ti \circ X_f = \pi_S \circ X_F \circ i,
\]

\( F \in C_M^\infty(U) \) is a local extension of \( f \) and \( \pi_S : TM|_S \to TS \) is the projection induced by \( TM|_S = B^\#((TS)^\circ) \oplus TS \) of \( TM|_S \).
(v) The symplectic leaves of \((S, \{\cdot, \cdot\}^S)\) are the connected components of \(S \cap \mathcal{L}\), with \(\mathcal{L}\) a symplectic leaf of \((M, \{\cdot, \cdot\})\). Any symplectic leaf of \((S, \{\cdot, \cdot\}^S)\) is a symplectic submanifold of the symplectic leaf of \((M, \{\cdot, \cdot\})\) that contains it.

(vi) Let \(\mathcal{L}_s\) and \(\mathcal{L}_s^S\) be the symplectic leaves of \((M, \{\cdot, \cdot\})\) and \((S, \{\cdot, \cdot\}^S)\) that contain \(s \in S\). Let \(\omega_{\mathcal{L}_s}\) and \(\omega_{\mathcal{L}_s^S}\) be their symplectic forms. Then \(B^\#(s)((T_sS)^\circ)\) is a symplectic subspace of \(T_s\mathcal{L}_s\) and

\[
B^\#(s)((T_sS)^\circ) = \left(T_s\mathcal{L}_s^S\right)^\omega_{\mathcal{L}_s}(s).
\]
(vii) Let $B_{S} \in \Lambda^{2}(S)$ be the Poisson tensor associated to $(S, \{\cdot, \cdot\}^{S})$. Then

$$B_{S}^\# = \pi_{S} \circ B_{S}^\# |_{S} \circ \pi_{S}^*,$$

where $\pi_{S}^* : \mathcal{T}^{*} S \to T^{*} M |_{S}$ is the dual of $\pi_{S} : \mathcal{T} M |_{S} \to TS$. 
$S \subset M$ is cosymplectic in $(M, \{\cdot, \cdot\})$ if and only if it satisfies the following two properties:

(i) $T_S S \cap T_S \mathcal{L}_s$ is a symplectic subspace of $(T_S \mathcal{L}_s, \omega_{\mathcal{L}_s}(s))$, where $\mathcal{L}_s$ is the symplectic leaf of $(M, \{\cdot, \cdot\})$ that contains $s \in S$;

(ii) $T_S S + T_S \mathcal{L}_s = T_S M$, for any $s \in S$. 
Dirac Formula

\((M, \{\cdot, \cdot\})\) \(n\)-dimensional Poisson, \(S\) a \(k\)-dimensional cosymplectic submanifold of \(M\). Can find coordinates \((\varphi^1, \ldots, \varphi^k, \psi_1, \ldots, \psi_{n-k})\) such that

\[
\begin{align*}
\bullet & \quad \varphi^1, \ldots, \varphi^k \text{ are } D\text{-invariant, } n - k = 2r \\
\bullet & \quad \{X_{\varphi^1}(s), \ldots, X_{\varphi^k}(s), X_{\psi_1}(s), \ldots, X_{\psi_{n-k}}(s)\} \text{ spans } T_s\mathcal{L}_s \\
\bullet & \quad \text{span}\{X_{\varphi^1}(s), \ldots, X_{\varphi^k}(s)\} = T_sS \cap T_s\mathcal{L}_s
\end{align*}
\]
\[
\{X_{\psi_1}(s), \ldots, X_{\psi_{n-k}}(s)\} \text{ is a basis of } \mathcal{B}^\#(s)((T_s S)^\circ)
\]

\[
\begin{pmatrix}
B_S & 0 \\
0 & C
\end{pmatrix}
\]

\[
C^{ij}(s) := \{\psi^i, \psi^j\}(s), i, j \in \{1, \ldots, n - k = 2r\}, \quad C = (C^{ij}) \text{ is invertible, } C^{-1} = (C_{ij})
\]
For any \( f, g \in C^\infty_{S,M}(V) \)

\[
X_f(s) = X_F(s) - \sum_{i,j=1}^{n-k} \{F, \psi^i\}(s)C_{ij}(s)X_{\psi^j}(s)
\]

and

\[
\{f, g\}^S(s) = \{F, G\}(s) - \sum_{i,j=1}^{n-k} \{F, \psi^i\}(s)C_{ij}(s)\{\psi^j, G\}(s),
\]

where \( F, G \in C^\infty_M(U) \) are arbitrary local extensions of \( f \) and \( g \) around \( s \in S \).
Dirac structures
Important remarks. (1) Counterexample

Associated smooth (co)distributions