On the geometry of coisotropic submanifolds of Poisson manifolds

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Abstract. We review basic results about Poisson manifolds and their coisotropic submanifolds. We show that the classical existence theorem for coisotropic embeddings of presymplectic manifolds can be extended to the case of Dirac manifolds. We give local normal forms describing these coisotropic embeddings.

1. Introduction

Coisotropic submanifolds of symplectic manifolds are important tools for the study of mechanical systems with symmetries (see for instance [**FHST89**] and references therein). In the mid-eighties, Libermann and Marle [**LM87**] extended the notion of a coisotropic submanifold to the general setting of Poisson structures. The interest in such objects has been growing due to the fact that they are closely related to various physical theories. For instance, coisotropic submanifolds of Poisson manifolds represent boundary conditions in various topological field theories (see [**CF04**]). They provide a suitable framework for solving the quantization problem of mechanical systems [**BW97, B00, B05, BGHHW, CF04, CF07**]. Basically, the quantization problem is the construction of a quantum theory for a given classical mechanical system. In the physics literature, coisotropic submanifolds appear under the name of first-class constraints. Here, we discuss only their geometric aspects.

It is known that any presymplectic manifold whose presymplectic form has constant rank can be coisotropically embedded into some symplectic manifold that consists of a tubular neighborhood of the zero section of the dual of its characteristic bundle together with an adapted symplectic structure [**G84**]. This result can be extended to the general setting of Dirac manifolds as observed by Cattaneo and Zambon [**CZ06**]. Here, we present a new proof of this result. We also give explicit expressions for coisotropic embeddings by using local normal forms of Dirac structures given in [**DW04, V06**].

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The article is organized as follows: In Sections 2 and 3, we review basic concepts and known facts about Poisson manifolds, coisotropic submanifolds of Poisson manifolds and Dirac manifolds. In Section 4, we discuss the existence and the uniqueness of coisotropic embeddings of Dirac manifolds and provide an example. In Section 5, we give local normal forms describing coisotropic embeddings of Dirac manifolds and conclude with some remarks.

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2. Preliminaries

2.1. Poisson structures.

Definition 2.1. Let M be a smooth finite-dimensional manifold. A *Poisson structure* on M is an $\mathbb{R}$-bilinear antisymmetric operation $\{\ ,\ \}$ on the space $C^\infty(M)$ of smooth real-valued functions on M satisfying the following identities:

$$\textit{Jacobi identity:}\qquad \{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0 \tag{1}$$

$$\textit{Leibniz identity:}\qquad \{f,gh\}=\{f,g\}h+g\{f,h\}, \tag{2}$$

for all $f,g,h\in C^\infty(M)$.

In other words, the space $C^\infty(M)$ is equipped with a Lie bracket $\{\ ,\ \}$ which, additionally, satisfies the Leibniz identity. The operation $\{\ ,\ \}$ is also called a *Poisson bracket*. A manifold M endowed with a Poisson structure is called a *Poisson manifold*.

Example 2.1. Every manifold admits a trivial Poisson structure given by $\{f,g\}=0$ for all functions f and g.

Example 2.2. Consider $M=\mathbb{R}^{2n}$ with its standard coordinates (x_i,y_i) and define a Poisson structure on it by setting

$$\{f,g\}=\sum_{i=1}^{n}\Big(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}-\frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\Big), \tag{3}$$

for all $f,g\in C^\infty(M)$. One can easily check that both the Jacobi identity and the Leibniz identity are satisfied for this bracket.

Example 2.3. By a *non-degenerate* Poisson structure on a $2n$-dimensional manifold M, we mean a bilinear operation which is related to a closed 2-form Ω of rank $2n$ as follows:

$$\{f,g\}=\Omega(X_f,X_g), \tag{4}$$

where X_f is the Hamiltonian vector field given by

$$i_{X_f}\Omega=-df,\quad \forall\ f\in C^\infty(M).$$

The Leibniz identity follows from $d(gh)=gdh+hdg$. While the Jacobi identity for the bracket (4) is equivalent to the fact $d\Omega=0$. In fact, Ω is called a *symplectic form*. Non-degenerate Poisson structures are exactly symplectic structures.

EXAMPLE 2.4. The dual $\mathfrak{g}^*$ of any finite-dimensional Lie algebra $(\mathfrak{g}, [\ ,\ ])$ admits a canonical Poisson structure, called *a Lie-Poisson* structure and defined by:

$$\{f, g\}(\alpha) = \langle \alpha, [df(\alpha), dg(\alpha)] \rangle, \quad \forall\ f, g \in C^\infty(\mathfrak{g}^*) \quad \forall \alpha \in \mathfrak{g}^*, \tag{5}$$

where $T_\alpha \mathfrak{g}^* \simeq \mathfrak{g}^*$, the terms $df(\alpha)$ and $dg(\alpha)$ are consider as elements of the Lie algebra $\mathfrak{g}$.

EXAMPLE 2.5. **Product of Poisson manifolds**. Let $(M_1, \{\ ,\ \}_1)$ and $(M_2, \{\ ,\ \}_2)$ be two Poisson manifolds. On the space of smooth real-valued functions on $M_1 \times M_2$, we define the following bracket:

$$\{f, g\}(x_1, x_2) = \{f_{x_2}, g_{x_2}\}_1(x_1) + \{f_{x_1}, g_{x_1}\}_2(x_2), \tag{6}$$

where

$$f_{x_1}(x_2) = f_{x_2}(x_1) = f(x_1, x_2),$$

for any $f \in C^\infty(M_1 \times M_2)$, $x_1 \in M_1$ and $x_2 \in M_2$. One can easily check that the above bracket is a Poisson bracket on $M_1 \times M_2$. It is called a *product Poisson structure.*

REMARK 2.1. Every Poisson structure $\{\ ,\ \}$ on M is uniquely characterized by an associated bivector field π defined as follows:

$$\pi(df, dg) = \{f, g\}, \qquad f, g \in C^\infty(M).$$

Conversely, a bivector field π on M defines a Poisson structure if and only if the Schouten bracket $[\pi, \pi]$ is zero (see [**V94, DZ05**]). In local coordinates $(x_1, \ldots, x_d)$, one has the expression:

$$\pi = \sum_{i<j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where the π_{ij} are smooth functions. The condition $[\pi, \pi] = 0$ means that

$$\sum_{s=1}^{d} \Big(\frac{\partial \pi_{ij}}{\partial x_s} \pi_{sk} + \frac{\partial \pi_{jk}}{\partial x_s} \pi_{si} + \frac{\partial \pi_{ki}}{\partial x_s} \pi_{sj} \Big) = 0. \tag{7}$$

Such a bivector field π is called a *Poisson tensor* on M. It determines a bundle morphism $\pi^\sharp : T^*M \to TM$ given by:

$$\langle \beta, \pi^\sharp \alpha \rangle = \pi(\alpha, \beta),$$

for all $\alpha, \beta \in T^*M$.

2.2. Coisotropic submanifolds. Let C be a submanifold of a Poisson manifold $(M, \{\ ,\ \})$. Consider the *vanishing ideal* $I_C = \{f \in C^\infty(M) \mid f_{|_C} = 0\}$.

DEFINITION 2.2. The submanifold C is *coisotropic* if I_C is closed under the Poisson bracket $\{\ ,\ \}$.

Now we review some known facts about coisotropic submanifolds.

a. Let π be the corresponding Poisson tensor on M and let N^*C be the conormal bundle of C, i.e.

$$N^*C(x) = \{\alpha_x \in T_x^*M \mid \langle \alpha_x, u \rangle = 0, \forall\ u \in T_xC\}.$$

Then C is a coisotropic submanifold of M if and only if

$$\pi^\sharp(N^*C) \subset TC.$$

b. Now suppose π defines a non-degenerate Poisson structure on M and let Ω be its corresponding symplectic form. Then the map $\pi^\sharp : N^*C \to \mathrm{Orth}_\Omega TC$ is an isomorphism, where $\mathrm{Orth}_\Omega TC$ is given by:

$$\mathrm{Orth}_\Omega TC = \{v \in TM \mid \Omega(u,v) = 0,\ \forall u \in TC\}.$$

In the non-degenerate case, C is coisotropic if and only if

$$\mathrm{Orth}_\Omega TC \subset TC.$$

c. There are coisotropic submanifolds of Poisson manifolds M whose intersections with symplectic leaves are not coisotropic. Under some transversality, any intersection with a symplectic leaf is coisotropic. More precisely, assume that C intersects cleanly the symplectic leaves of M, i.e.

$$C \cap F \text{ is a manifold and } T(C \cap F) = TC \cap TF,$$

for every symplectic leaf F of M, then $C \cap F$ is coisotropic. We should mention that reductions of coisotropic submanifolds were studied in Poisson geometry [**MR86**].

2.3. Coisotropic embeddings of presymplectic manifolds. Recall that a *presymplectic form* on a smooth manifold M is a closed 2-form (not necessarily non-degenerate). A manifold equipped with a presymplectic form is called a *presymplectic manifold*. When the rank of the presymplectic form ω is constant then its kernel $\mathrm{Ker}\omega$ defines an involutive distribution. It determines a foliation called a *characteristic foliation*. We have the following coisotropic embedding result:

Proposition 2.1. [**G84**] *Given any presymplectic manifold (C, ω) whose presymplectic form has constant rank, there exists a symplectic form on a tubular neighborhood of the zero section of the dual bundle K^* of $K = \mathrm{Ker}\omega$ such that C can be coisotropically embedded into this neighborhood.*

Moreover, any two coisotropic embeddings of (C, ω) are locally equivalent (up to symplectomorphism).

A proof of this proposition, which uses methods in [**We81**], can be found in [**G84**]. Another proof is given in [**OP05**].

3. Dirac manifolds

3.1. Basic definitions and examples. The *Courant bracket* [**C90**] on the space of smooth sections of the vector bundle $TM \oplus T^*M \to M$ is a natural extension of the Lie bracket of vector fields on M. Precisely, it is defined by:

$$[(X_1,\alpha_1),\ (X_2,\alpha_2)] = \Big([X_1,X_2],\ \mathcal{L}_{X_1}\alpha_2 - i_{X_2}d\alpha_1\Big), \tag{8}$$

for all (X_1,α_1) and $(X_2,\alpha_2) \in \Gamma(TM \oplus T^*M)$, where $[X_1,X_2]$ is the usual Lie bracket of vector fields, $\mathcal{L}_{X_1}\alpha_2$ the Lie derivative of the 1-form α_2 along X_1.

Definition 3.1. [**C90**] A *Dirac structure* on a smooth d-dimensional manifold M is a rank d subbundle L of $TM \oplus T^*M \to M$ whose space of smooth sections is closed under the Courant bracket and such that:

$$\langle (X_1,\alpha_1),\ (X_2,\alpha_2)\rangle = \frac{1}{2}\Big(\alpha_1(X_2) + \alpha_2(X_1)\Big) = 0, \tag{9}$$

for all smooth sections (X_1,α_1) and $(X_2,\alpha_2) \in \Gamma(TM \oplus T^*M)$. A manifold equipped with a Dirac structure is called a *Dirac manifold*.

EXAMPLE 3.1. [**C90**] Presymplectic structures are in one-to-one correspondence with Dirac structures L with $L \cap (\{0\} \times T^*M) = \{0\}$. Indeed, given any presymplectic form Ω, its graph

$$L_\Omega = \{(X, i_X\Omega) \mid X \in TM\}$$

defines a Dirac structure on M with $L \cap (\{0\} \times T^*M) = \{0\}$. Conversely, any Dirac structure on M satisfying this condition determines a presymplectic form on M.

EXAMPLE 3.2. There is a one-to-one correspondence between Poisson structures and Dirac structures L on M satisfying $L \cap (TM \times \{0\}) = \{0\}$. The Dirac structure associated with a Poisson structure π is:

$$L_\pi = \{(\pi^\sharp\alpha, \alpha) \mid \alpha \in T^*M\}.$$

3.2. Presymplectic foliation of a Dirac manifold. Let (M, L) be a Dirac manifold and $\mathrm{pr} : L \to TM$ the natural projection of L onto TM. Then $\mathcal{D} = \mathrm{pr}(L)$ is an involutive distribution which determines a foliation, i.e. a partition of M into connected and immersed submanifolds, called leaves. The dimensions of the leaves may vary. But each leaf of M carries a presymplectic form defined at the point $x \in M$ as follows:

$$\theta_x(X_1, X_2) = \alpha_1(X_2), \tag{10}$$

where α_1 is any element in T_x^*M satisfying $(X_1, \alpha_1) \in L_x$ and $(X_2, \alpha_2) \in L_x$. Equation (9) implies that (10) is independent of the choice of the covector α_1 satisfying $(X_1, \alpha_1) \in L_x$. The 2-forms θ_x fit together into a smooth leafwise 2-form θ. Using the fact that $\Gamma(L)$ is closed under the Courant bracket, one sees that θ is closed on each leaf.

3.3. Gauge transformations and Dirac maps. Now we will describe some methods which allow to construct new Dirac structures starting from a given one.

Gauge transformations. Let L be a Dirac structure on M and B a closed 2-form, then

$$\tau_B(L) := \{(X, \alpha + i_X B) \mid (X, \alpha) \in L\} \tag{11}$$

defines a Dirac structure on M [**SW01**]. The operation τ_B is called a *gauge transformation associated with* B. Basically, this operation modifies L by adding the pull-back of B to the presymplectic form on each leaf.

Dirac maps. Let (M_1, L_1) and (M_2, L_2) be two Dirac manifolds. A smooth map $f : M_1 \to M_2$ is called a *forward Dirac map* if L_1 and L_2 are related as follows:

$$Ł_2 = \{(df(Y),\ \alpha) \mid Y \in TM_1,\ \alpha \in T^*M_2 \text{ and } (Y,\ df^*(\alpha)) \in L_1\}. \tag{12}$$

We simplicity, we will write $L_2 = f_*L_1$.

A smooth map $g : M_1 \to M_2$ is called a *backward Dirac map* if

$$L_1 = \{(Y,\ dg^*(\alpha)) \mid Y \in TM_1,\ \alpha \in T^*M_2 \text{ and } (dg(Y),\ \alpha) \in L_2\}. \tag{13}$$

In this case, we write $L_1 = g^*L_2$.

Observe that g^*L_2 is a well-defined subbundle of TM_1 but it is not necessarily smooth. While f_*L_1 may not be well-defined. However, when g is a submersion, g^*L_2 is always a smooth Dirac structure.

Example 3.3. Suppose M_1 and M_1 are Poisson manifolds. Then $f : M_1 \to M_2$ is a forward Dirac map if and only if its graph $Gr(f)$ is a coisotropic submanifold of the product Poisson manifold $M_1 \times M_2$.

Example 3.4. Let L be a Dirac structure on M. We consider a presymplectic leaf F of M, equipped with a Dirac structure L_ω associated with the presymplectic form ω. The inclusion map $\iota : F \hookrightarrow M$ is both a backward map and a forward map [**BR03**].

4. Coisotropic embeddings of Dirac manifolds

4.1. Existence Theorem. In this section, we are interested in the following question: Given a Dirac manifold C, does there exist a coisotropic embedding of (C, L) into some adapted Poisson manifold? First all, if $K = L \cap (TC \times \{0\}) = \{0\}$ then C is a Poisson manifold. In this case, there is nothing to prove. Now suppose that K is non-trivial. We aim to generalize Gotay's embedding theorem for presymplectic manifolds to the setting of Dirac structures. We obtain:

Theorem 4.1. *Let C be a smooth finite-dimensional manifold endowed with a Dirac structure L such that the vector bundle $K = L \cap (TC \times \{0\})$ has constant rank. Then, there exist a neighborhood U of the zero section in the dual K^* of K and a Poisson structure π_U on U such that C can be coisotropically embedded into (U, π_U).*

Proof. Since K has constant rank, it determines a foliation $\mathcal{K}$ on C. Consider the zero section embedding $\iota : C \hookrightarrow K^*$ and identify K with its first projection onto TC. Let $\Gamma : TC \to K$ be a bundle map such that $\Gamma^2 = \Gamma$. Then $TC = \mathrm{Ker}(\Gamma) \oplus K$. The map Γ can be considered as a K-valued 1-form, which is preserved by the flow generated by any vector field X tangent to the foliation $\mathcal{K}$, i.e. $\mathcal{L}_X \Gamma = 0$ (see details about the local expressions of Γ in Section 5). There is a corresponding 1-form $\theta_{_\Gamma}$ on K^* defined as follows:

$$\theta_{_\Gamma}(\alpha)(Z) = \langle \alpha, \Gamma \circ dp(Z) \rangle, \tag{14}$$

where $p : K^* \to C$ is the canonical projection of K^* onto C, $\alpha \in K^*$, and $Z \in T_\alpha K^*$.

Furthermore, there is an induced foliation on K^* whose leaves are the connected components of the preimages under p of the leaves of $\mathcal{K}$. Let $\mathrm{Vert} \subset TK^*$ be the tangent distribution to that induced foliation. On a small tubular neighborhood U of C in K^*, the 2-form $\theta_{_\Gamma}$ is non-degenerate on the vertical subspaces Vert_α (see the local expressions in Section 5). Moreover, it determines a field of horizontal subspaces:

$$\mathrm{Hor}_\Gamma(\alpha) = \{Y_\alpha \in T_\alpha K^* \mid d\theta_{_\Gamma}(X, Y) = 0, \ \forall \ X \in \mathrm{Vert}_\alpha\}.$$

Choose a real number $\lambda > 0$ and define the Dirac structure L_λ on K^* by setting

$$L_\lambda = \tau_{d(\lambda\theta_{_\Gamma})}(p^* L),$$

where $\tau_{d(\lambda\theta_{_\Gamma})}$ is the gauge transformation associated with $d(\lambda\theta_{_\Gamma})$. Then, L_λ determines a Dirac structure which is a Poisson structure on K^* provided that λ is small enough. There follows the result.

□

REMARK 4.1. Cattaneo and Zambon [**CZ06**] obtained a result similar to Theorem 4.1. Their description is different from the one given in the above proof and it is implicit.

A priori, the Poisson structure corresponding to L_λ depends on the choices of Γ and λ. Suppose that $L \cap (TC \times \{0\})$ defines a regular and *simple* foliation $\mathcal{K}$. In this case, L is said to be *reducible*. It induces a Poisson structure on the quotient manifold $C/\mathcal{K}$, which is Hausdorff and smooth. Under the additional assumption that L is reducible, one can show all coisotropic embeddings of (C, L) are locally equivalent (up to Dirac maps). The proof, which is quite technical, will not be presented here.

4.2. Example. Consider $C = S^3 \times \mathfrak{so}^*(3)$, i.e. the product of the sphere $S^3 \subset \mathbb{R}^4$ with the dual of the Lie algebra $\mathfrak{so}(3)$. Here, $\mathfrak{so}^*(3)$ is endowed with its canonical Lie-Poisson structure Λ, which is inherited from the Lie algebra structure on $\mathfrak{so}(3)$, while S^3 is equipped with its contact form $\alpha = i^*\sigma$, where $\sigma = 1/2\sum(p_i dq_i - q_i dp_i)$ is the Liouville form on $\mathbb{R}^4$ and $i : S^3 \hookrightarrow \mathbb{R}^4$ is the inclusion map. Consider the Dirac structure L on C whose space of sections is spanned by the sections of the form
$$(X,\ i_X d\alpha), \quad (\Lambda^\sharp\beta,\ \beta),$$
where $X \in \mathfrak{X}(S^3)$ and $\beta \in \Omega^1(\mathfrak{so}^*(3))$. Clearly, $(S^3 \times \mathfrak{so}^*(3), L)$ is a reducible Dirac manifold. Its characteristic foliation is determined by the orbits of the flow of the Reeb vector field $\mathbf{Z}$ on S^3 , i.e. the circles of the Hopf fibration. One can see that $(S^3 \times \mathfrak{so}^*(3), L)$ embeds coisotropically into $M = \mathbb{R} \times S^3 \times \mathfrak{so}^*(3)$ with the product Poisson structure $\omega^{-1} \oplus \Lambda$, where $\omega = d(e^t\alpha)$ is the symplectization of the contact structure on S^3.

5. Local normal forms

In this section, we give the local expressions of the above Dirac structure L_λ.

PROPOSITION 5.1. [**DW04, V06**] *Let L be a Dirac structure on C such that $K = L \cap (TC \oplus \{0\})$ has constant rank. Given any point $q \in C$, there are coordinates $(x_1, \dots, x_r, x'_1, \dots, x'_s, y_1, \dots, y_n)$ defined on an open neighborhood W of q in C such that $L_{|W}$ is spanned by sections of the form*

$$\begin{aligned}
F_j &= \Big(\frac{\partial}{\partial y_j},\ 0\Big) \\
G_i &= \Big(\frac{\partial}{\partial x_i} + \sum_j R_i^j(x,x')\frac{\partial}{\partial x'_j},\ \ \sum_j \omega_i^j(x,x')dx_j\Big) \\
G'_k &= \Big(\sum_\ell \pi_{k\ell}(x,x')\frac{\partial}{\partial x'_\ell},\ \ dx'_k - \sum_l R_\ell^k(x,x')dx_l\Big),
\end{aligned}$$

where the $R_i^j(x,x')$ and $\pi_{k\ell}(x,x')$ vanish at $(x,x') = (0,0)$.

Proposition 5.1 is a consequence of the local normal forms given in [**DW04**]. A complete proof of this result can be found in [**V06**]. In fact, this proposition ensures the existence of a local coordinate system $(x,y) = (x_1, \dots, x_m, y_1, \dots, y_n)$ defined on a neighborhood W of any point q in C such the intersections of W with the leaves of $\mathcal{K}$ are given by the equations
$$x_1 = c_1,\ \dots,\ x_m = c_m, \quad \text{with } c_i \in \mathbb{R},$$

and $L_{|W}$ is determined by sections of the form

$$F_j = \Big(\frac{\partial}{\partial y_j}, 0\Big) \quad \text{and} \quad E = \Big(\sum_{i=1}^m h_i \frac{\partial}{\partial x_i}(x), \ \ \sum_{i=1}^m \alpha_i(x) dx_i\Big),$$

where the functions h_i and α_i depend only on the x-coordinates. We have the local expressions:

$$\Gamma = \sum_{i=1}^n \Big(dy_i + \Gamma_i^j dx_j\Big) \otimes \frac{\partial}{\partial y_i}. \tag{15}$$

By a simple computation, one gets

$$\theta_{\Gamma} = \sum_{i=1}^n z_i (dy_i + \Gamma_i^j dx_j),$$

where $(x_1, \dots, x_m, y_1, \dots, y_n, z_1, \dots, z_n)$ are canonical coordinates defined on an open neighborhood of V_q of q in K^* such that the z_i's are linear on the fibers of K^*. One gets

$$d\theta_{\Gamma} \quad = \quad \sum_{i,j} dz_i \wedge (dy_i + \Gamma_i^j dx_j) + \sum_{i,j,\ell} z_i \frac{\partial \Gamma_i^j}{\partial x_\ell} dx_\ell \wedge dx_j + \sum_{i,j,k} z_i \frac{\partial \Gamma_i^j}{\partial y_k} dy_k \wedge dx_j.$$

Since $\mathcal{L}_X \Gamma = 0$, for any vector field X tangent to the foliation $\mathcal{K}$, it follows

$$d\theta_{\Gamma} = \sum_{i,j,k,u} dz_i \wedge (dy_i + \Gamma_i^j dx_j) + \sum_{i,j,\ell} z_i \frac{\partial \Gamma_i^j}{\partial x_\ell} dx_\ell \wedge dx_j.$$

Now, we consider local smooth sections of L having the expression:

$$h = \Big(\sum_i h_i(x) \frac{\partial}{\partial x_i}, \ \ \sum_i \alpha_i(x) dx_i\Big) \quad \text{with} \quad (\alpha_1, \dots, \alpha_m) \neq (0, \dots, 0).$$

We want to find all vector fields of the form

$$Y = \sum_i h_i(x) \frac{\partial}{\partial x_i} + \sum_j \mu_j(x,y,z) \frac{\partial}{\partial y_j} + \sum_k \nu_k(x,y,z) \frac{\partial}{\partial z_k}$$

satisfying the equations:

$$d\theta_{\Gamma}\Big(Y, \frac{\partial}{\partial y_k}\Big) = d\theta_{\Gamma}\Big(Y, \frac{\partial}{\partial z_k}\Big) = 0,$$

for all $k \in \{1, \dots, n\}$. Solving these equations one gets

$$Y = \sum h_i \left(\frac{\partial}{\partial x_i} - \sum_j \Gamma_j^i \frac{\partial}{\partial y_j}\right).$$

We set

$$\mathbf{X}_i = \frac{\partial}{\partial x_i} - \sum_j \Gamma_j^i \frac{\partial}{\partial y_j}.$$

We have obtained the following proposition:

PROPOSITION 5.2. *On the open neighborhood V_q, the space of local sections of L_λ is spanned by elements of the form*

$$\begin{aligned} e_i &= \Big(\frac{\partial}{\partial z_i}, \quad dy_i + \sum_j \Gamma_i^j dx_j\Big), \\ e'_i &= \Big(\frac{\partial}{\partial y_i}, \quad -dz_i\Big) \\ e'' &= \Big(\sum_i h_i(x)\mathbf{X}_i, \quad \sum_i (\alpha_i(x) + \lambda\beta_i)dx_i\Big), \end{aligned}$$

where the β_i are components of $i_Y d\theta_\Gamma$.

These local expressions of L_λ show that it is a maximally isotropic subbundle with respect to the symmetric operation $\langle\ ,\ \rangle$, i.e. Equation (9) is satisfied. When λ is small enough, $\alpha_i + \lambda\beta_i \neq 0$. Moreover, one gets

$$L_\lambda \cap (TV_q \oplus \{0\}) = \{0\}. \tag{16}$$

Hence L_λ determines a Poisson structure on V_q.

□

REMARK 5.1. **a.** Local expressions for coisotropic embeddings of presympletic manifolds were given in [**OP05**]. Our local normal forms generalize their formulas.

b. As announced above (Remark 4.1), the uniqueness of coisotropic embeddings of Dirac manifolds, up to local Dirac maps that coincide with the identity on C, can be established when L is a reducible Dirac structure on C. Another particular case where the leafwise presymplectic form on C has constant rank is treated in [**CZ06**]. However, the general case is still an open problem.

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