“On Lunar and Solar Periodicities of Earthquakes.” By ARTHUR SCHUSTER, F.R.S. Received May 18,—Read June 17, 1897.
Stochastic Process

Definition (stochastic process)

A stochastic process is sequence of indexed random variables denoted as $Z(\omega, t)$ where $\omega$ belongs to a sample space and $t$ belongs to an index set.

From here on out, we will simply write a stochastic process (or time series) as $\{Z_t\}$ (dropping the $\omega$).

Notation

For the time units $t_1, t_2, \ldots, t_n$, denote the $n$-dimensional distribution function of $Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}$ as

$$F_{Z_{t_1}, \ldots, Z_{t_n}} (x_1, \ldots, x_n) = P(Z_{t_1} \leq x_1, \ldots, Z_{t_n} \leq x_n)$$

Example

Let $Z_1, \ldots, Z_n \overset{iid}{\sim} \mathcal{N}(0, 1)$. Then $F_{Z_{t_1}, \ldots, Z_{t_n}} (x_1, \ldots, x_n) = \prod_{j=1}^n \Phi(x_j)$.
Definition (stochastic process)

A stochastic process is sequence of indexed random variables denoted as $Z(\omega, t)$ where $\omega$ belongs to a sample space and $t$ belongs to an index set.

From here on out, we will simply write a stochastic process (or time series) as $\{Z_t\}$ (dropping the $\omega$).

Notation

For the time units $t_1, t_2, \ldots, t_n$, denote the $n$-dimensional distribution function of $Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}$ as

$$F_{Z_{t_1}, \ldots, Z_{t_n}}(x_1, \ldots, x_n) = P(Z_{t_1} \leq x_1, \ldots, Z_{t_n} \leq x_n)$$

Example

Let $Z_1, \ldots, Z_n \sim \mathcal{N}(0, 1)$. Then $F_{Z_{t_1}, \ldots, Z_{t_n}}(x_1, \ldots, x_n) = \prod_{j=1}^{n} \Phi(x_j)$. 
**Stochastic Process**

**Definition (stochastic process)**

A stochastic process is a sequence of indexed random variables denoted as $Z(\omega, t)$ where $\omega$ belongs to a sample space and $t$ belongs to an index set.

From here on out, we will simply write a stochastic process (or time series) as $\{Z_t\}$ (dropping the $\omega$).

**Notation**

For the time units $t_1, t_2, \ldots, t_n$, denote the $n$-dimensional distribution function of $Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}$ as

$$F_{Z_{t_1}, \ldots, Z_{t_n}}(x_1, \ldots, x_n) = P(Z_{t_1} \leq x_1, \ldots, Z_{t_n} \leq x_n)$$

**Example**

Let $Z_1, \ldots, Z_n \sim \mathcal{N}(0, 1)$. Then $F_{Z_{t_1}, \ldots, Z_{t_n}}(x_1, \ldots, x_n) = \prod_{j=1}^{n} \Phi(x_j)$. 
A stochastic process is sequence of indexed random variables denoted as $Z(\omega, t)$ where $\omega$ belongs to a sample space and $t$ belongs to an index set.

From here on out, we will simply write a stochastic process (or time series) as $\{Z_t\}$ (dropping the $\omega$).

Notation

For the time units $t_1, t_2, \ldots, t_n$, denote the $n$-dimensional distribution function of $Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}$ as

$$F_{Z_{t_1}, \ldots, Z_{t_n}}(x_1, \ldots, x_n) = P(Z_{t_1} \leq x_1, \ldots, Z_{t_n} \leq x_n)$$

Example

Let $Z_1, \ldots, Z_n \overset{iid}{\sim} N(0, 1)$. Then

$$F_{Z_{t_1}, \ldots, Z_{t_n}}(x_1, \ldots, x_n) = \prod_{j=1}^{n} \Phi(x_j).$$
Definition (Strongly Stationarity (aka Strictly, aka Completely))

A time series \( \{x_t\} \) is strongly stationary if any collection

\[
\{x_{t_1}, x_{t_2}, \ldots, x_{t_n}\}
\]

has the same joint distribution as the time shifted set

\[
\{x_{t_1} + h, x_{t_2} + h, \ldots, x_{t_n} + h\}
\]

Strong stationarity implies the following:

- All marginal distributions are equal, i.e. \( P(Z_s \leq c) = P(Z_t \leq c) \) for all \( s, t, \) and \( c \). This is what is called “first-order stationary”.

- The covariance of \( Z_s \) and \( Z_t \) (if exists) is shift-independent, i.e. \( E(Z_s + h, Z_t + h) = E(Z_s, Z_t) \) for any choice of \( h \).

Strong stationarity typically assumes too much. This leads us to the weaker assumption of \textit{weak} stationarity.
Definition (Strongly Stationarity (aka Strictly, aka Completely))

A time series \( \{x_t\} \) is strongly stationary if any collection

\[
\{x_{t_1}, x_{t_2}, \ldots, x_{t_n}\}
\]

has the same joint distribution as the time shifted set

\[
\{x_{t_1} + h, x_{t_2} + h, \ldots, x_{t_n} + h\}
\]

Strong stationarity implies the following:

- All marginal distributions are equal, i.e. \( P(Z_s \leq c) = P(Z_t \leq c) \) for all \( s, t, \) and \( c \). This is what is called “first-order stationary”.

- The covariance of \( Z_s \) and \( Z_t \) (if exists) is shift-independent, i.e.
  \[
  E(Z_s + h, Z_t + h) = E(Z_s, Z_t)
  \]
  for any choice of \( h \).

Strong stationarity typically assumes too much. This leads us to the weaker assumption of weak stationarity.
Stationarity

**Definition (Strongly Stationarity (aka Strictly, aka Completely))**

A time series \( \{x_t\} \) is strongly stationary if any collection

\[
\{x_{t_1}, x_{t_2}, \ldots, x_{t_n}\}
\]

has the same joint distribution as the time shifted set

\[
\{x_{t_1+h}, x_{t_2+h}, \ldots, x_{t_n+h}\}
\]

Strong stationarity implies the following:

- All marginal distributions are equal, i.e. \( P(Z_s \leq c) = P(Z_t \leq c) \) for all \( s, t, \) and \( c \). *This is what is called “first-order stationary”*. 

- The covariance of \( Z_s \) and \( Z_t \) (if exists) is shift-independent, i.e. 
  \[
  E(Z_s + h, Z_t + h) = E(Z_s, Z_t)
  \]
  for any choice of \( h \).

Strong stationarity typically assumes too much. This leads us to the weaker assumption of *weak* stationarity.
Definition (Strongly Stationarity (aka Strictly, aka Completely))
A time series \( \{x_t\} \) is strongly stationary if any collection
\[
\{x_{t_1}, x_{t_2}, \ldots, x_{t_n}\}
\]
has the same joint distribution as the time shifted set
\[
\{x_{t_1}+h, x_{t_2}+h, \ldots, x_{t_n}+h\}
\]

Strong stationarity implies the following:
- All marginal distributions are equal, i.e. \( P(Z_s \leq c) = P(Z_t \leq c) \) for all \( s, t, \) and \( c \). \textit{This is what is called “first-order stationary”}.
- The covariance of \( Z_s \) and \( Z_t \) (if exists) is shift-independent, i.e.
  \[
  \text{E}(Z_s + h, Z_t + h) = \text{E}(Z_s, Z_t) \text{ for any choice of } h.
  \]

Strong stationarity typically assumes too much. This leads us to the weaker assumption of \textit{weak} stationarity.
Mean Function

Definition (Mean Function)

The mean function of a time series \{Z_t\} (if it exists) is given by

\[
\mu_t = \mathbb{E}(Z_t) = \int_{-\infty}^{\infty} x f_t(x) \, dx
\]

Example (Mean of an iid MA(q))

Let \(a_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)\) and

\[Z_t = a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \cdots + \theta_q a_{t-q}\]

then

\[
\mu_t = \mathbb{E}(Z_t) = \mathbb{E}(a_t) + \theta_1 \mathbb{E}(a_{t-1}) + \theta_2 \mathbb{E}(a_{t-2}) + \cdots + \theta_q \mathbb{E}(a_{t-q}) \equiv 0
\]

(free of the time variable \(t\))
Mean Function

Definition (Mean Function)

The mean function of a time series \( \{Z_t\} \) (if it exists) is given by

\[
\mu_t = E(Z_t) = \int_{-\infty}^{\infty} x f_t(x) \, dx
\]

Example (Mean of an iid MA(\(q\)))

Let \( a_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2) \) and

\[
Z_t = a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \cdots + \theta_q a_{t-q}
\]

then

\[
\mu_t = E(Z_t) = E(a_t) + \theta_1 E(a_{t-1}) + \theta_2 E(a_{t-2}) + \cdots + \theta_q E(a_{t-q}) \equiv 0
\]

(free of the time variable \( t \))
Mean Function

Definition (Mean Function)
The mean function of a time series \( \{Z_t\} \) (if it exists) is given by

\[
\mu_t = \mathbb{E}(Z_t) = \int_{-\infty}^{\infty} x f_t(x) \, dx
\]

Example (Mean of an iid MA(\(q\)))
Let \( a_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2) \) and

\[
Z_t = a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \cdots + \theta_q a_{t-q}
\]

then

\[
\mu_t = \mathbb{E}(Z_t) = \mathbb{E}(a_t) + \theta_1 \mathbb{E}(a_{t-1}) + \theta_2 \mathbb{E}(a_{t-2}) + \cdots + \theta_q \mathbb{E}(a_{t-q}) = 0
\]

(free of the time variable \( t \))
Definition (White Noise)

**White noise** is a collection of uncorrelated random variables with constant mean and variance.

- **Notation**
  \[ a_t \sim WN(0, \sigma^2) \] — white noise with mean zero and variance \( \sigma^2 \)

- **IID WN**
  If \( a_s \) is independent of \( a_t \) for all \( s \neq t \), then \( w_t \sim IID(0, \sigma^2) \)

- **Gaussian White Noise \( \Rightarrow \) IID**
  Suppose \( a_t \) is normally distributed.
  uncorrelated + normality \( \Rightarrow \) independent
  Thus it follows that \( a_t \sim IID(0, \sigma^2) \) (a stronger assumption).
**White Noise**

**Definition (White Noise)**

*White noise* is a collection of uncorrelated random variables with constant mean and variance.

- **Notation**
  
  \[ a_t \sim \text{WN}(0, \sigma^2) \]  
  — white noise with mean zero and variance \( \sigma^2 \)

- **IID WN**
  
  If \( a_s \) is independent of \( a_t \) for all \( s \neq t \), then \( w_t \sim \text{IID}(0, \sigma^2) \)

- **Gaussian White Noise \( \Rightarrow \) IID**

  Suppose \( a_t \) is normally distributed.

  *uncorrelated + normality \( \Rightarrow \) independent*

  Thus it follows that \( a_t \sim \text{IID}(0, \sigma^2) \) (a stronger assumption).
White Noise

Definition (White Noise)

White noise is a collection of uncorrelated random variables with constant mean and variance.

- **Notation**
  \[ a_t \sim \text{WN}(0, \sigma^2) \] — white noise with mean zero and variance \( \sigma^2 \)

- **IID WN**
  If \( a_s \) is independent of \( a_t \) for all \( s \neq t \), then \( w_t \sim \text{IID}(0, \sigma^2) \)

- **Gaussian White Noise \( \Rightarrow \) IID**
  Suppose \( a_t \) is normally distributed.
  uncorrelated + normality \( \Rightarrow \) independent
  Thus it follows that \( a_t \sim \text{IID}(0, \sigma^2) \) (a stronger assumption).
**White Noise**

**Definition (White Noise)**

White noise is a collection of uncorrelated random variables with constant mean and variance.

- **Notation**
  
  \( a_t \sim \text{WN}(0, \sigma^2) \) — white noise with mean zero and variance \( \sigma^2 \)

- **IID WN**

  If \( a_s \) is independent of \( a_t \) for all \( s \neq t \), then \( w_t \sim \text{IID}(0, \sigma^2) \)

- **Gaussian White Noise \( \Rightarrow \) IID**

  Suppose \( a_t \) is normally distributed.

  uncorrelated + normality \( \Rightarrow \) independent

  Thus it follows that \( a_t \sim \text{IID}(0, \sigma^2) \) (a stronger assumption).
Now your turn!

Mean of a Random Walk with Drift

Suppose $Z_0 = 0$, and for $t > 0$, $Z_t = \delta + Z_{t-1} + a_t$ where $\delta$ is a constant and $a_t \sim \text{WN}(0, \sigma^2)$. What is the mean function of $Z_t$? (That is compute $E[Z_t]$.)
Now your turn!

Mean of a Random Walk with Drift

Suppose $Z_0 = 0$, and for $t > 0$, $Z_t = \delta + Z_{t-1} + a_t$ where $\delta$ is a constant and $a_t \sim \text{WN}(0, \sigma^2)$. What is the mean function of $Z_t$? (That is compute $E[Z_t]$.)

Note that

\[ Z_t = \delta + Z_{t-1} + a_t \]
\[ = 2\delta + Z_{t-2} + a_t + a_{t-1} \quad (Z_{t-1} = \delta + Z_{t-2} + a_{t-1}) \]
\[ \vdots \]
\[ = \delta t + \sum_{j=1}^{t} a_j \]

Therefore we see

\[ \mu_t = E[Z_t] = E[\delta t] + E\left[\sum_{j=1}^{t} a_j\right] = \delta t + \sum_{j=1}^{t} E[a_j] = \delta t \]
The Covariance Function

Definition (Covariance Function)
The covariance function of a random sequence \( \{Z_t\} \) is

\[
\gamma(s, t) = \text{cov}(Z_s, Z_t) = \mathbb{E} [(Z_s - \mu) (Z_t - \mu)]
\]

So in particular, we have

\[
\text{var}(Z_t) = \text{cov}(Z_t, Z_t) = \gamma(t, t) = \mathbb{E} [(Z_t - \mu)^2]
\]

Also note that \( \gamma(s, t) = \gamma(t, s) \) since \( \text{cov}(Z_s, Z_t) = \text{cov}(Z_t, Z_s) \).

Example (Covariance Function of White Noise)
Let \( a_t \sim \text{WN}(0, \sigma^2) \). By the definition of white noise, we have

\[
\gamma(s, t) = \mathbb{E}(a_s, a_t) = \begin{cases} 
\sigma^2, & s = t \\
0, & s \neq t 
\end{cases}
\]
The Covariance Function

**Definition (Covariance Function)**

The covariance function of a random sequence \( \{Z_t\} \) is

\[
\gamma(s, t) = \text{cov}(Z_s, Z_t) = \mathbb{E}[(Z_s - \mu_s)(Z_t - \mu_t)]
\]

So in particular, we have

\[
\text{var}(Z_t) = \text{cov}(Z_t, Z_t) = \gamma(t, t) = \mathbb{E}[(Z_t - \mu_t)^2]
\]

Also note that \( \gamma(s, t) = \gamma(t, s) \) since \( \text{cov}(Z_s, Z_t) = \text{cov}(Z_t, Z_s) \).

**Example (Covariance Function of White Noise)**

Let \( a_t \sim \text{WN}(0, \sigma^2) \). By the definition of white noise, we have

\[
\gamma(s, t) = \mathbb{E}(a_s, a_t) = \begin{cases} 
\sigma^2, & s = t \\
0, & s \neq t 
\end{cases}
\]
The Covariance Function

**Definition (Covariance Function)**

The covariance function of a random sequence \( \{Z_t\} \) is

\[
\gamma(s, t) = \text{cov}(Z_s, Z_t) = \mathbb{E}[(Z_s - \mu_s)(Z_t - \mu_t)]
\]

So in particular, we have

\[
\text{var}(Z_t) = \text{cov}(Z_t, Z_t) = \gamma(t, t) = \mathbb{E}[(Z_t - \mu_t)^2]
\]

Also note that \( \gamma(s, t) = \gamma(t, s) \) since \( \text{cov}(Z_s, Z_t) = \text{cov}(Z_t, Z_s) \).

**Example (Covariance Function of White Noise)**

Let \( a_t \sim \text{WN}(0, \sigma^2) \). By the definition of white noise, we have

\[
\gamma(s, t) = \mathbb{E}(a_s, a_t) = \begin{cases} 
\sigma^2, & s = t \\
0, & s \neq t
\end{cases}
\]
Example (Covariance Function of MA(1))

Let $a_t \sim WN(0, \sigma^2)$ and $Z_t = a_t + \theta a_{t-1}$ (where $\theta$ is a constant), then

\[
\gamma(s, t) = \text{cov}(a_t + \theta a_{t-1}, a_s + \theta a_{s-1}) \\
= \text{cov}(a_t, a_s) + \theta \text{cov}(a_t, a_{s-1}) + \\
+ \theta \text{cov}(a_{t-1}, a_s) + \theta^2 \text{cov}(a_{t-1}, a_{s-1})
\]

If $s = t$, then

\[
\gamma(s, t) = \gamma(t, t) = \sigma^2 + \theta^2 \sigma^2 = (\theta^2 + 1)\sigma^2
\]

If $s = t - 1$ or $s = t + 1$, then

\[
\gamma(s, t) = \gamma(t, t + 1) = \gamma(t, t - 1) = \theta\sigma^2
\]

So all together we have

\[
\gamma(s, t) = \begin{cases} 
(\theta^2 + 1)\sigma^2, & \text{if } s = t \\
\theta\sigma^2, & \text{if } |s - t| = 1 \\
0, & \text{else}
\end{cases}
\]
Example (Covariance Function of MA(1))

Let \( a_t \sim \text{WN}(0, \sigma^2) \) and \( Z_t = a_t + \theta a_{t-1} \) (where \( \theta \) is a constant), then

\[
\gamma(s, t) = \text{cov}(a_t + \theta a_{t-1}, a_s + \theta a_{s-1})
\]
\[
= \text{cov}(a_t, a_s) + \theta \text{cov}(a_t, a_{s-1}) + \theta \text{cov}(a_{t-1}, a_s) + \theta^2 \text{cov}(a_{t-1}, a_{s-1})
\]

If \( s = t \), then
\[
\gamma(s, t) = \gamma(t, t) = \sigma^2 + \theta^2 \sigma^2 = (\theta^2 + 1) \sigma^2
\]

If \( s = t - 1 \) or \( s = t + 1 \), then
\[
\gamma(s, t) = \gamma(t, t + 1) = \gamma(t, t - 1) = \theta \sigma^2
\]

So all together we have

\[
\gamma(s, t) = \begin{cases} 
(\theta^2 + 1) \sigma^2, & \text{if } s = t \\
\theta \sigma^2, & \text{if } |s - t| = 1 \\
0, & \text{else}
\end{cases}
\]
Example (Covariance Function of MA(1))

Let $a_t \sim \text{WN}(0, \sigma^2)$ and $Z_t = a_t + \theta a_{t-1}$ (where $\theta$ is a constant), then

$$
\gamma(s, t) = \text{cov}(a_t + \theta a_{t-1}, a_s + \theta a_{s-1}) \\
= \text{cov}(a_t, a_s) + \theta \text{cov}(a_t, a_{s-1}) + \\
+ \theta \text{cov}(a_{t-1}, a_s) + \theta^2 \text{cov}(a_{t-1}, a_{s-1})
$$

If $s = t$, then

$$
\gamma(s, t) = \gamma(t, t) = \sigma^2 + \theta^2 \sigma^2 = (\theta^2 + 1)\sigma^2
$$

If $s = t - 1$ or $s = t + 1$, then

$$
\gamma(s, t) = \gamma(t, t + 1) = \gamma(t, t - 1) = \theta \sigma^2
$$

So all together we have

$$
\gamma(s, t) = \begin{cases} 
(\theta^2 + 1)\sigma^2, & \text{if } s = t \\
\theta \sigma^2, & \text{if } |s - t| = 1 \\
0, & \text{else}
\end{cases}
$$
Let $a_t \sim \text{WN}(0, \sigma^2)$ and $Z_t = a_t + \theta a_{t-1}$ (where $\theta$ is a constant), then

\[
\gamma(s, t) = \text{cov}(a_t + \theta a_{t-1}, a_s + \theta a_{s-1})
\]
\[
= \text{cov}(a_t, a_s) + \theta \text{cov}(a_t, a_{s-1}) + \theta \text{cov}(a_{t-1}, a_s) + \theta^2 \text{cov}(a_{t-1}, a_{s-1})
\]

If $s = t$, then

\[
\gamma(s, t) = \gamma(t, t) = \sigma^2 + \theta^2 \sigma^2 = (\theta^2 + 1)\sigma^2
\]

If $s = t - 1$ or $s = t + 1$, then

\[
\gamma(s, t) = \gamma(t, t + 1) = \gamma(t, t - 1) = \theta \sigma^2
\]

So all together we have

\[
\gamma(s, t) = \begin{cases}
(\theta^2 + 1)\sigma^2, & \text{if } s = t \\
\theta \sigma^2, & \text{if } |s - t| = 1 \\
0, & \text{else}
\end{cases}
\]
Example (Covariance Function of MA(1))

Let $a_t \sim \text{WN}(0, \sigma^2)$ and $Z_t = a_t + \theta a_{t-1}$ (where $\theta$ is a constant), then

$$
\gamma(s, t) = \text{cov}(a_t + \theta a_{t-1}, a_s + \theta a_{s-1})
$$

$$
= \text{cov}(a_t, a_s) + \theta \text{cov}(a_t, a_{s-1}) +
$$

$$
+ \theta \text{cov}(a_{t-1}, a_s) + \theta^2 \text{cov}(a_{t-1}, a_{s-1})
$$

If $s = t$, then

$$
\gamma(s, t) = \gamma(t, t) = \sigma^2 + \theta^2 \sigma^2 = (\theta^2 + 1)\sigma^2
$$

If $s = t - 1$ or $s = t + 1$, then

$$
\gamma(s, t) = \gamma(t, t + 1) = \gamma(t, t - 1) = \theta \sigma^2
$$

So all together we have

$$
\gamma(s, t) = \begin{cases} 
(\theta^2 + 1)\sigma^2, & \text{if } s = t \\
\theta \sigma^2, & \text{if } |s - t| = 1 \\
0, & \text{else}
\end{cases}
$$
Now your turn!

Covariance Function of a Random Walk with Drift

Suppose $Z_0 = 0$, and for $t > 0$, $Z_t = \delta + Z_{t-1} + a_t$ where $\delta$ is a constant and $a_t \sim \text{WN}(0, \sigma^2)$. What is the covariance function of $Z_t$? (That is compute $\text{cov}(Z_s, Z_t)$.)
Covariance Function of a Random Walk with Drift

Suppose $Z_0 = 0$, and for $t > 0$, $Z_t = \delta + Z_{t-1} + a_t$ where $\delta$ is a constant and $a_t \sim \text{WN}(0, \sigma^2)$. What is the covariance function of $Z_t$? (That is compute $\text{cov}(Z_s, Z_t)$.)

From the representation

$$Z_t = \delta t + \sum_{j=1}^{t} a_j$$

We see

$$\gamma(s, t) = \text{cov}(Z_s, Z_t) = \text{cov} \left( \sum_{j=1}^{s} a_j, \sum_{k=1}^{t} a_k \right)$$

$$= \sum_{j=1}^{s} \sum_{k=1}^{t} \text{cov} (a_j, a_k) = \min(s, t) \sigma^2$$
The Autocorrelation Function

Definition (Correlation Function)

The correlation function of a random sequence \( \{Z_t\} \) is

\[
\rho(s, t) = \frac{\gamma(s, t)}{\sqrt{\gamma(s, s)\gamma(t, t)}}
\]

Cauchy-Schwartz inequality \( \Rightarrow |\rho(s, t)| \leq 1 \)
The Autocorrelation Function

Definition (Correlation Function)

The correlation function of a random sequence \( \{Z_t\} \) is

\[
\rho(s, t) = \frac{\gamma(s, t)}{\sqrt{\gamma(s, s)\gamma(t, t)}}
\]

Cauchy-Schwartz inequality \( \Rightarrow |\rho(s, t)| \leq 1 \)
Weak Stationarity

Definition (Weak Stationarity)

A weakly stationary time series is a finite variance process that

(i) has a mean function, $\mu$, that is constant (so it doesn’t depend on $t$);

(ii) has a covariance function, $\gamma(s, t)$, that depends on $s$ and $t$ only through the difference $|s - t|$.

From now on when we say stationary, we mean weakly stationary.

- For stationary processes, the mean function is $\mu$ (no $t$).
- Since $\gamma(s, t) = \gamma(s + h, t + h)$ for stationary processes, we have $\gamma(s, t) = \gamma(s - t, 0)$.

Therefore we can view the covariance function as a function of only one variable ($h = s - t$); this is the \textit{autocovariance function} $\gamma(h)$. 
Weak Stationarity

Definition (Weak Stationarity)
A weakly stationary time series is a finite variance process that

(i) has a mean function, \( \mu \), that is constant (so it doesn’t depend on \( t \));

(ii) has a covariance function, \( \gamma(s, t) \), that depends on \( s \) and \( t \) only through
the difference \( |s - t| \).

From now on when we say stationary, we mean weakly stationary.

- For stationary processes, the mean function is \( \mu \) (no \( t \)).
- Since \( \gamma(s, t) = \gamma(s + h, t + h) \) for stationary processes, we have
  \( \gamma(s, t) = \gamma(s - t, 0) \).
  Therefore we can view the covariance function as a function of only one variable \( (h = s - t) \); this is the *autocovariance function* \( \gamma(h) \).
Weak Stationarity

Definition (Weak Stationarity)

A weakly stationary time series is a finite variance process that

(i) has a mean function, $\mu$, that is constant (so it doesn’t depend on $t$);

(ii) has a covariance function, $\gamma(s, t)$, that depends only on $s$ and $t$ through the difference $|s - t|$.

From now on when we say stationary, we mean weakly stationary.

- For stationary processes, the mean function is $\mu$ (no $t$).

- Since $\gamma(s, t) = \gamma(s + h, t + h)$ for stationary processes, we have
  $\gamma(s, t) = \gamma(s - t, 0)$.

Therefore we can view the covariance function as a function of only one variable ($h = s - t$); this is the **autocovariance function** $\gamma(h)$.
Weak Stationarity

Definition (Weak Stationarity)

A weakly stationary time series is a finite variance process that

(i) has a mean function, $\mu$, that is constant (so it doesn’t depend on $t$);

(ii) has a covariance function, $\gamma(s, t)$, that depends on $s$ and $t$ only through the difference $|s - t|$.

From now on when we say stationary, we mean weakly stationary.

- For stationary processes, the mean function is $\mu$ (no $t$).
- Since $\gamma(s, t) = \gamma(s + h, t + h)$ for stationary processes, we have $\gamma(s, t) = \gamma(s - t, 0)$.
  Therefore we can view the covariance function as a function of only one variable ($h = s - t$); this is the *autocovariance function* $\gamma(h)$.
Outline

1. §2.1: Stationarity
2. §2.2: Autocovariance and Autocorrelation Functions
3. §2.4: White Noise
4. R: Random Walk
5. Homework 1b
Autocovariance and Autocorrelation Functions

Definition (Autocovariance Function)

The autocovariance function of a stationary time series is

\[ \gamma_h = \gamma(h) = E \left[ (Z_{t+h} - \mu)(Z_t - \mu) \right] \]

(for any value of \( t \)).

Definition (Autocorrelation Function)

The autocorrelation function of a stationary time series is

\[ \rho_h = \rho(h) = \frac{\gamma(h)}{\gamma(0)} \]
Autocovariance and Autocorrelation Functions

Definition (Autocovariance Function)

The autocovariance function of a stationary time series is

$$\gamma_h = \gamma(h) = E \left[ (Z_{t+h} - \mu)(Z_t - \mu) \right]$$

(for any value of $t$).

Definition (Autocorrelation Function)

The autocorrelation function of a stationary time series is

$$\rho_h = \rho(h) = \frac{\gamma(h)}{\gamma(0)}$$
\section*{Properties of the Autocovariance/Autocorrelation Functions}

- $\gamma_0 = \text{var}(Z_t)$; $\rho_0 = 1$
- $|\gamma_k| \leq \gamma_0$; $|\rho_k| \leq 1$
- $\gamma_k = \gamma_{-k}$; $\rho_k = \rho_{-k}$
- $\gamma_k$ and $\rho_k$ are positive semidefinite (p.s.d.), i.e.
  \[ \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i - t_j|} \geq 0 \]
  and
  \[ \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \rho_{|t_i - t_j|} \geq 0 \]
  for any set of times $t_1, \ldots, t_n$ and any real numbers $\alpha_1, \ldots, \alpha_n$.

\textbf{Proof.}

Define $X = \sum_{i=1}^{n} \alpha_i Z_{t_i}$, then

\[ 0 \leq \text{var}(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \text{cov}(Z_{t_i}, Z_{t_j}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i - t_j|} \]
Properties of the Autocovariance/Autocorrelation Functions

- $\gamma_0 = \text{var}(Z_t)$; $\rho_0 = 1$
- $|\gamma_k| \leq \gamma_0$; $|\rho_k| \leq 1$
- $\gamma_k = \gamma_{-k}$; $\rho_k = \rho_{-k}$
- $\gamma_k$ and $\rho_k$ are positive semidefinite (p.s.d.), i.e.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i-t_j|} \geq 0$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \rho_{|t_i-t_j|} \geq 0$$

for any set of times $t_1, \ldots, t_n$ and any real numbers $\alpha_1, \ldots, \alpha_n$.

Proof.

Define $X = \sum_{i=1}^{n} \alpha_i Z_{t_i}$, then

$$0 \leq \text{var}(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \text{cov}(Z_{t_i}, Z_{t_j}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i-t_j|}$$
Properties of the Autocovariance/Autocorrelation Functions

- $\gamma_0 = \text{var}(Z_t)$; $\rho_0 = 1$
- $|\gamma_k| \leq \gamma_0$; $|\rho_k| \leq 1$
- $\gamma_k = \gamma_{-k}$; $\rho_k = \rho_{-k}$
- $\gamma_k$ and $\rho_k$ are positive semidefinite (p.s.d.), i.e.

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i-t_j|} \geq 0
\]

and

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \rho_{|t_i-t_j|} \geq 0
\]

for any set of times $t_1, \ldots, t_n$ and any real numbers $\alpha_1, \ldots, \alpha_n$.

**Proof.**

Define $X = \sum_{i=1}^{n} \alpha_i Z_{t_i}$, then

\[
0 \leq \text{var}(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \text{cov}(Z_{t_i}, Z_{t_j}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i-t_j|}
\]
Properties of the Autocovariance/Autocorrelation Functions

- \( \gamma_0 = \text{var}(Z_t); \rho_0 = 1 \)
- \( |\gamma_k| \leq \gamma_0; |\rho_k| \leq 1 \)
- \( \gamma_k = \gamma_{-k}; \rho_k = \rho_{-k} \)
- \( \gamma_k \) and \( \rho_k \) are positive semidefinite (p.s.d.), i.e.

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i - t_j|} \geq 0
\]

and

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \rho_{|t_i - t_j|} \geq 0
\]

for any set of times \( t_1, \ldots, t_n \) and any real numbers \( \alpha_1, \ldots, \alpha_n \).

**Proof.**

Define \( X = \sum_{i=1}^{n} \alpha_i Z_{t_i} \), then

\[
0 \leq \text{var}(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \text{cov}(Z_{t_i}, Z_{t_j}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i - t_j|}
\]
Properties of the Autocovariance/Autocorrelation Functions

- \( \gamma_0 = \text{var}(Z_t); \rho_0 = 1 \)
- \( |\gamma_k| \leq \gamma_0; |\rho_k| \leq 1 \)
- \( \gamma_k = \gamma_{-k}; \rho_k = \rho_{-k} \)
- \( \gamma_k \) and \( \rho_k \) are positive semidefinite (p.s.d.), i.e.

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i-t_j|} \geq 0
\]

and

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \rho_{|t_i-t_j|} \geq 0
\]

for any set of times \( t_1, \ldots, t_n \) and any real numbers \( \alpha_1, \ldots, \alpha_n \).

Proof.

Define \( X = \sum_{i=1}^{n} \alpha_i Z_{t_i} \), then

\[
0 \leq \text{var}(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \text{cov}(Z_{t_i}, Z_{t_j}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \gamma_{|t_i-t_j|}
\]
Outline

1. §2.1: Stationarity
2. §2.2: Autocovariance and Autocorrelation Functions
3. §2.4: White Noise
4. R: Random Walk
5. Homework 1b
White Noise

White noise, denoted by \( a_t \sim WN(0, \sigma^2) \), is by definition a weakly stationary process with autocovariance function

\[
\gamma_k = \begin{cases} 
\sigma^2, & k = 0 \\
0, & k \neq 0 
\end{cases}
\]

and autocorrelation function

\[
\rho_k = \begin{cases} 
1, & k = 0 \\
0, & k \neq 0 
\end{cases}
\]

Not all white noise is boring like iid \( \mathcal{N}(0, \sigma^2) \). For example, stationary GARCH processes can have all the properties of white noise.
White Noise

White noise, denoted by $a_t \sim WN(0, \sigma^2)$, is by definition a weakly stationary process with autocovariance function

$$\gamma_k = \begin{cases} \sigma^2, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

and autocorrelation function

$$\rho_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

Not all white noise is boring like iid $\mathcal{N}(0, \sigma^2)$. For example, stationary GARCH processes can have all the properties of white noise.
Outline

1. §2.1: Stationarity
2. §2.2: Autocovariance and Autocorrelation Functions
3. §2.4: White Noise
4. R: Random Walk
5. Homework 1b
Random Walk Simulation in R

Simulate the random walk \( Z_t = 0.2 + Z_{t-1} + a_t \) where \( a_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2) \).

Hint: use the representation \( Z_t = 0.2t + \sum_{j=1}^{t} a_j \! \! \text{!} \)

```r
> w = rnorm(200, 0, 1)
> x = cumsum(w)
> wd = w + 0.2
> xd = cumsum(wd)
> plot.ts(xd, ylim=c(-5, 55))
> lines(x)
> lines(0.2*(1:200), lty="dashed")
```

![Graph of simulated random walk](image)
Outline

1. §2.1: Stationarity
2. §2.2: Autocovariance and Autocorrelation Functions
3. §2.4: White Noise
4. R: Random Walk
5. Homework 1b
Read the following sections from the textbook

- §2.5: Estimation of the Mean, Autocovariances, and Autocorrelations
- §2.6: Moving Average and Autoregressive Representations of Time Series Processes

Do the following exercise.

- Prove that a time series of iid random variables is strongly stationary.