Decay for the Maxwell system on Schwarzschild/Kerr backgrounds

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Outline.

1. The black hole stability problem
2. The wave equation
3. The Maxwell system
Vacuum Einstein Equations

Determine manifolds $M$ with Lorentzian metric $g$ and vanishing Ricci curvature:

$$R_{ij} = 0$$

- cosmological constant $\Lambda = 0$
- System of nonlinear wave equations
- 10 equations for 10 unknowns $g_{ij}$
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- cosmological constant $\Lambda = 0$
- System of nonlinear wave equations
- 10 equations for 10 unknowns $g_{ij}$
- 4 relations between equations $\nabla^\alpha R_{\alpha\beta} = 0 \implies$ 6 independent equations
- 4 degrees of (gauge) freedom = choice of coordinates
The Minkowski space-time

\[ M = \mathbb{R} \times \mathbb{R}^3 \]

\[ ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \]
The Schwarzschild spacetime

- In polar coordinates, the metric is

\[
 ds^2 = -(1 - \frac{2M}{r}) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\omega^2
\]
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- (unique family of) spherically symmetric asymptotically flat stationary black holes parametrized by the mass \( M \)
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- Singularity at $r = 0$, but not at $r = 2M$ (the event horizon). Let

$$\upsilon = t + r^*, \quad r^* = r + 2M \log(r - 2M)$$

The metric in the $(r, \upsilon, \omega)$ coordinates becomes

$$ds^2 = -(1 - \frac{2M}{r})d\upsilon^2 + 2d\upsilon dr + r^2d\omega^2$$

and can be extended past the event horizon.
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- Singularity at \( r = 0 \), but not at \( r = 2M \) (the event horizon). Let \( v = t + r^* \), \( r^* = r + 2M \log(r - 2M) \)

  The metric in the \((r, v, \omega)\) coordinates becomes
  \[ ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2d\omega^2 \]

  and can be extended past the event horizon.

- Our preferred set of coordinates will be \((\tilde{t}, r, \omega)\), where \( \tilde{t} = v - \mu(r) \)

  for a suitable smooth function \( \mu \) equal to \( r^* \) for \( r > \frac{5M}{2} \).
The Kerr spacetime

- In Boyer-Lindquist coordinates, the metric is

\[ ds^2 = g_{tt} dt^2 + g_{t\phi} dt d\phi + g_{rr} dr^2 + g_{\phi\phi} d\phi^2 + g_{\theta\theta} d\theta^2 \]

where

\[ g_{tt} = -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2}, \quad g_{t\phi} = -2a \frac{2Mr \sin^2 \theta}{\rho^2}, \quad g_{rr} = \frac{\rho^2}{\Delta} \]

\[ g_{\phi\phi} = \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2} \sin^2 \theta, \quad g_{\theta\theta} = \rho^2 \]

with \( \Delta = r^2 - 2Mr + a^2 \), \( \rho^2 = r^2 + a^2 \cos^2 \theta \).
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with $\Delta = r^2 - 2Mr + a^2$, $\rho^2 = r^2 + a^2 \cos^2 \theta$.

- (unique family of) axis symmetric asymptotically flat stationary black holes parametrized by the mass $M$ and angular momentum $aM$.

- The Kerr metric is a small perturbation $O(a)$ of the Schwarzschild metric; assume $a$ small from now on.
The stability question

- The initial value problem: given a 3D manifold $N$ with Riemannian metric $g_0$ and a symmetric two-tensor $k_0$ satisfying constraint equations, find a 4D manifold $M$ with Lorentzian metric $g$ satisfying Einstein’s Equations and $g_0$ is the restriction of $g$ to $N$, $k_0$ is the second fundamental form.

What if the initial data is close to the initial data of the solutions described above?

Theorem (Christodoulou-Klainerman, Lindblad-Rodnianski)

The Minkowski space-time is nonlinearly stable as a solution to the vacuum Einstein equation.

Open Problem

Is the (exterior of) the Kerr family asymptotically stable?
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The linearized problem:

1. **Scalar waves:**

   \[ \nabla^\alpha \nabla_\alpha \phi = 0 \]

   Fairly well understood
The linearized problem:

1. Scalar waves:
   \[ \nabla^\alpha \nabla_\alpha \phi = 0 \]
   Fairly well understood

2. Maxwell’s equations (spin 1)
   \[ \nabla^\alpha \nabla_\alpha V^\gamma = 0, \quad \nabla^\gamma V_\gamma = 0 \]
   - 4 equations for 4 unknowns (electromagnetic potential),
   - one relation between equations \( \implies \) 3 independent equations + 1 degree of freedom (gauge choice)
   - Main topic of the talk
The linearized problem:

3. Linearized gravity (spin 2)

\[ \nabla^\alpha \nabla_\alpha h_{\gamma\delta} = 2g^{\alpha\beta}g^{\mu\nu}R_{\gamma\mu\delta\alpha}h_{\nu\beta} \]

\[ \nabla^\gamma h_{\gamma\delta} - \frac{1}{2}g^{\alpha\beta}\nabla_\delta h_{\alpha\beta} = 0 \]

- 10 equations for 10 unknowns
- 4 relations between equations \(\implies\) 6 independent equations + 4 degrees of freedom (gauge choice)
- Good research direction

M. Tohaneanu (JHU)  Maxwell on Schwarzschild/Kerr  July 2012
Two ways to measure decay

Consider the equation

$$\Box_g u = f, \quad u(0) = u_0, \quad Ku(0) = u_1$$

- Local energy spaces

$$\|u\|_{LE} = \sup_k \|\langle r \rangle^{-\frac{1}{2}} u\|_{L^2(\mathbb{R} \times A_k)}, \quad A_k = \{|x| \approx 2^k\} \times \mathbb{R}$$

We also define its $H^1$ counterpart, as well as the dual norm

$$\|u\|_{LE^1} = \|\nabla u\|_{LE} + \|\langle r \rangle^{-1} u\|_{LE} \quad \|f\|_{LE^*} = \sum_k \|\langle r \rangle^{\frac{1}{2}} f\|_{L^2(\mathbb{R} \times A_k)}$$
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- Pointwise decay
Minkowski

- Local energy decay:

\[ \| \nabla u \|_{L^\infty L^2} + \| u \|_{LE^1} \lesssim \| \nabla u_0 \|_{L^2} + \| u_1 \|_{L^2} + \| f \|_{LE^* + L^1 L^2} \]
Minkowski

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Morawetz, Strauss, Kenig-Ponce-Vega, Smith-Sogge, Keel-Smith-Sogge, Rodnianski
Minkowski

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\[ \|\nabla u\|_{L^\infty L^2} + \|u\|_{L^1E^1} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2} + \|f\|_{L^1L^1} \]

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Also true for small perturbations: Alinhac, Metcalfe-Sogge, Metcalfe-Tataru
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  Also true for small perturbations: Alinhac, Metcalfe-Sogge, Metcalfe-Tataru

- Pointwise decay (homogeneous, compactly supported data):
  \[ |u(t, x)| \lesssim \frac{1}{\langle t \rangle \langle t - r \rangle^N} \]

  Not true for small perturbations!
Local energy decay:

\[
\|\nabla u\|_{L^\infty L^2} + \|u\|_{L^1 E^1} \leq \|\nabla u_0\|_{H^1} + \|u_1\|_{H^1} + \|f\|_{L^1 E^* + L^1 L^2}
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Laba-Soffer, Blue-Soffer, Blue-Sterbenz, Dafermos-Rodnianski, Marzuola-Metcalfe-Tataru-T. (Schwarzschild)
Schwarzschild/Kerr

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Schwarzschild/Kerr

- Local energy decay:

\[ \|\nabla u\|_{L^\infty} + \|u\|_{LE^1} \lesssim \|\nabla u_0\|_{H^1} + \|u_1\|_{H^1} + \|f\|_{LE^*+L^1} \]

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- Pointwise decay (homogeneous, compactly supported data):

\[ |u(\tilde{t}, x)| \lesssim \frac{1}{\langle \tilde{t} \rangle \langle \tilde{t} - r \rangle^2} \]

Finster-Kamran-Smoller-Yau, Dafermos-Rodnianski, Andersson-Blue, Luk, Tataru, Metcalfe-Tataru-T. (Kerr)
Maxwell on Schwarzschild/Kerr
Schwarzschild/Kerr

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Wald, Kay-Wald, Blue-Sterbenz, Dafermos-Rodnianski, Luk, Donninger-Schlag-Soffer, Tataru, Metcalfe-Tataru-T. (Schwarzschild)
Schwarzschild/Kerr

- Local energy decay:

\[ ||\nabla u||_{L^\infty L^2} + ||u||_{LE^1} \lesssim ||\nabla u_0||_{H^1} + ||u_1||_{H^1} + ||f||_{LE^* + L^1 L^2} \]

Laba-Soffer, Blue-Soffer, Blue-Sterbenz, Dafermos-Rodnianski, Marzuola-Metcalfe-Tataru-T. (Schwarzschild)
Dafermos-Rodnianski, Tataru-T., Andersson-Blue, Wunsch-Zworski (Kerr with |a| \ll M)
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Finster-Kamran-Smoller-Yau, Dafermos-Rodnianski, Andersson-Blue, Luk, Tataru, Metcalfe-Tataru-T. (Kerr)
Theorem (Metcalf-Tataru-T.)

(Price’s Law for nonstationary space times). Assume that higher order uniform energy bounds and local energy decay hold. Then the following pointwise decay bounds hold for smooth localized data:

\[
|u(t, x)| \lesssim \frac{1}{\langle t \rangle \langle t - |x| \rangle^2} \|\nabla u(0)\|_{H^m}
\]

\[
|\partial_t u(t, x)| \lesssim \frac{1}{\langle t \rangle \langle t - |x| \rangle^3} \|\nabla u(0)\|_{H^m}
\]

\[
|\nabla u(t, x)| \lesssim \frac{1}{\langle r \rangle \langle t - |x| \rangle^3} \|\nabla u(0)\|_{H^m}
\]
Sketch of proof

Use the vector fields:

\[ Z = (\partial_\alpha, \Omega = x_i \partial_j - x_j \partial_i, S = t \partial_t + x \partial_x) \]

Assume that

\[ g = m + g_{sr} + g_{lr} \]

where \( m \) stands for the Minkowski metric, \( g_{lr} \) is a stationary long range spherically symmetric component with coefficients \( \approx \frac{1}{r} \) and \( g_{sr} \) is a short range component with coefficients \( \approx \frac{1}{r^2} \).

Through a change of coordinates and conjugation, we can write

\[ \Box_g u = \Box u + \frac{1}{r^3} (\Omega^2 u + u) + \frac{1}{r^2} (\nabla^2 u + \nabla u) \quad (2.1) \]
Sketch of proof

Use the fundamental solution of the 1D wave equation and the local energy estimates to obtain a first estimate

\[ |u| \lesssim \frac{\ln (t-r)}{r(t-r)^{1/2}} \]

Plug into (2.1), obtain better estimates near the cone. Use Sobolev embeddings (associated to \( S \)) to improve decay inside. Can push the estimate to

\[ |u| \lesssim \frac{1}{t(t-r)} \]

Final step is to use the cancellation due to the presence of derivatives on the worst behaved term.
The Maxwell system

Electromagnetic field \( F = \) anti symmetric two form on \((M, g)\).

1. Via differential forms:

\[
dF = 0, \quad d \ast F = 0
\]

2. Using covariant differentiation:

\[
\nabla^\alpha F_{\alpha \beta} = 0, \quad \nabla_{[\gamma} F_{\alpha \beta]} = 0
\]

3. Using electromagnetic potential \( A \), \( F = dA \):

\[
\nabla^\alpha \nabla_\alpha A_\beta = 0, \quad \nabla^\alpha A_\alpha = 0 \quad \text{(gauge condition)}
\]

4. Expressed in a reference frame (Newman-Penrose formalism)
Each component $F_{ij}$ satisfies

$$\Box F_{ij} = 0$$

Therefore local energy and pointwise decay hold just like in the previous section.
Schwarzschild

There are stationary solutions!

\[ F = \frac{Q}{4\pi} d\omega_{S^2}, \quad F = \frac{Q^*}{4\pi} r^{-2} dr \wedge dt \]

\( Q \) is the electric charge, \( Q^* \) is the magnetic charge. Both need to be 0 in order to have decay!
Local energy decay

Theorem (Metcalfe-Sterbenz-Tataru-T.)

Local energy decay holds for charge free solutions in Schwarzschild:

\[ \|F\|_{L^\infty L^2} + \|F\|_{LE} \lesssim \|F(0)\|_{H^1} \]

Also inhomogeneous and higher order (Lie derivatives) versions of this hold, as long as no charges are introduced with the inhomogeneous terms.
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Idea of proof: Let \((l, n, e_A, e_B)\) be the null frame

\[ \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \partial_u, \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \partial_v, e_A, e_B, \]

and define

\[ \phi_{-A} = F(l, e_A), \phi_0 = F(l, n), \phi^*_0 = *F(e_A, e_B), \phi_{+A} = F(n, e_A) \]

Then \(\phi_0, \phi^*_0\) satisfy the scalar wave equation (with potential) and can use previous theory to get local energy decay. We can then use the coupling relations to transfer estimates.
Theorem (Metcalfe-Sterbenz-Tataru-T.)

The following peeling estimates hold for each component:

\[
|\phi_{-\mathcal{A}}| \lesssim \frac{1}{\langle r \rangle \langle t - r \rangle^3}
\]

\[
|\phi_0| + |\phi_0^*| \lesssim \frac{1}{\langle r \rangle \langle t \rangle \langle t - r \rangle^2}
\]

\[
|\phi_{+\mathcal{A}}| \lesssim \frac{1}{\langle r \rangle \langle t \rangle^2 \langle t - r \rangle}
\]

The \(\phi_{-\mathcal{A}}\) is the worst behaved because its \(u\)-derivative does not appear in the coupling relations!
Pointwise decay

**Theorem (Metcalfe-Sterbenz-Tataru-T.)**

The following peeling estimates hold for each component:

\[
|\phi_{-A}| \lesssim \frac{1}{\langle r \rangle \langle t - r \rangle^3}
\]

\[
|\phi_0| + |\phi^*_0| \lesssim \frac{1}{\langle r \rangle \langle t \rangle \langle t - r \rangle^2}
\]

\[
|\phi_{+A}| \lesssim \frac{1}{\langle r \rangle \langle t \rangle^2 \langle t - r \rangle}
\]

The \(\phi_{-A}\) is the worst behaved because its \(u\)-derivative does not appear in the coupling relations!

Finster-Smoller, Blue
The Newman-Penrose frame in Kerr is \((l, n, m, \bar{m})\) with

\[
l = \frac{1}{\Delta}(r^2 + a^2, +\Delta, 0, a)
\]

\[
n = \frac{1}{2|\rho|^2}(r^2 + a^2, -\Delta, 0, a)
\]

\[
m = \frac{1}{\rho \sqrt{2}}(ia \sin \theta, 0, 1, i/\sin \theta)
\]

with

\[
\rho = r + ia \cos \theta, \quad \Delta = r^2 - 2Mr + a^2
\]
Teukolski’s equations

The Maxwell field components in the NP frame are denoted by

\[ \phi_1 = F(l, m), \quad \phi_0 = \frac{1}{2} (F(l, n) + F(\bar{m}, m)) \quad \phi_{-1} = F(\bar{m}, n) \]

Normalization:

\[ \Phi_1 = \phi_1, \quad \Phi_0 = \sqrt{2} \bar{\rho} \phi_0, \quad \Phi_{-1} = 2 \bar{\rho}^2 \phi_{-1} \]

Teukolski: equations for \( \phi’ \)’s are uncoupled: \( \Psi_2 = -M\bar{\rho}^{-3} \)

\[
\left[ (\nabla^\mu + \Gamma^\mu)(\nabla_\mu + \Gamma_\mu) - 4\Psi_2 \right] \Phi_1 = 0 \\
\left[ \nabla^\mu \nabla_\mu - 2\Psi_2 \right] \Phi_0 = 0 \\
\left[ (\nabla^\mu - \Gamma^\mu)(\nabla_\mu - \Gamma_\mu) - 4\Psi_2 \right] \Phi_{-1} = 0
\]

\[ \Gamma = \left( -\frac{1}{\rho^2} \left[ \frac{M(r^2 - a^2)}{\Delta} - \rho \right], -\frac{1}{\rho^2} (r - M), 0, -\frac{1}{\rho^2} \left[ \frac{a(r - M)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right] \right) \]
Bianchi identities in NP frame:

Renormalize

\[
L = l, \quad N = -\frac{2\rho^2}{\Delta} n, \quad M = \sqrt{2}\rho m
\]

and observe that \(L\) and \(N\) commute with \(M, \tilde{M}\). Then

\[
\begin{align*}
(\tilde{M} + \cot \theta - \frac{ia \sin \theta}{\bar{\rho}}) \Phi_1 &= \left( L + \frac{1}{\bar{\rho}} \right) \Phi_0 \\
(\tilde{M} + \frac{ia \sin \theta}{\bar{\rho}}) \Phi_0 &= \left( L - \frac{1}{\bar{\rho}} \right) \Phi_{-1} \\
(M + \cot \theta - \frac{ia \sin \theta}{\bar{\rho}}) \Phi_{-1} &= -\Delta \left( N + \frac{1}{\bar{\rho}} \right) \Phi_0 \\
(M + \frac{ia \sin \theta}{\bar{\rho}}) \Phi_0 &= -\Delta \left( N + \frac{2(r - M)}{\Delta} - \frac{1}{\bar{\rho}} \right) \Phi_1
\end{align*}
\]