The Boundary value problems for second order elliptic operators satisfying Carleson condition

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Open problems
Dirichlet, Neumann and Regularity boundary value problems

Let $L = \text{div} \ A \nabla u$ be a second order elliptic operator with bounded measurable coefficients $A = (a_{ij})$ on a Lipschitz domain $\Omega$. That is there is $\Lambda > 0$ such that

$$\Lambda^{-1} |\xi|^2 \leq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2.$$  

(Matrix $A$ does not have to be symmetric).
Dirichlet, Neumann and Regularity boundary value problems

Let \( L = \text{div} \ A \nabla u \) be a second order elliptic operator with bounded measurable coefficients \( A = (a_{ij}) \) on a Lipschitz domain \( \Omega \). That is there is \( \Lambda > 0 \) such that

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(Matrix \( A \) does not have to be symmetric).
Let $\Gamma(.)$ be a collection of nontangential cones with vertices a boundary points $Q \in \partial \Omega$. We define the non-tangential maximal function at $Q$ relative to $\Gamma$ by

$$N(u)(Q) = \sup_{X \in \Gamma(Q)} |u(X)|.$$ 

We also consider a weaker version of this object

$$\tilde{N}(u)(Q) = \sup_{X \in \Gamma(Q)} \left( \delta(X)^{-n} \int_{B_{\delta(X)/2}(X)} |u(Y)|^2 \, dY \right)^{\frac{1}{2}}.$$
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Definition
Let $1 < p \leq \infty$. The Dirichlet problem with data in $L^p(\partial \Omega, d\sigma)$ is solvable (abbreviated $(D)_p$) if for every $f \in C(\partial \Omega)$ the weak solution $u$ to the problem $Lu = 0$ with continuous boundary data $f$ satisfies the estimate

$$\|N(u)\|_{L^p(\partial \Omega, d\sigma)} \lesssim \|f\|_{L^p(\partial \Omega, d\sigma)}.$$ 

The implied constant depends only the operator $L$, $p$, and the Lipschitz constant of the domain.
The Boundary value problems for second order elliptic operators satisfying Carleson condition

Formulation of boundary value problems

$L^p$ Neumann problem

Definition

Let $1 < p < \infty$. The Neumann problem with boundary data in $L^p(\partial\Omega)$ is solvable (abbreviated $(N)_p$), if for every $f \in L^p(\partial\Omega) \cap C(\partial\Omega)$ such that $\int_{\partial\Omega} f d\sigma = 0$ the weak solution $u$ to the problem

$$\begin{cases}
Lu = 0 & \text{in } \Omega \\
A\nabla u \cdot \nu = f & \text{on } \partial\Omega
\end{cases}$$

satisfies

$$\|\tilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

Again, the implied constant depends only the operator $L$, $p$, and the Lipschitz constant of the domain. Here $\nu$ is the outer normal to the boundary $\partial\Omega$. 

$\tilde{N}$ is the Neumann operator defined by $\tilde{N}(v) = \int_{\partial\Omega} A\nabla v \cdot \nu d\sigma.$
Regularity problem

Definition
Let $1 < p < \infty$. The regularity problem with boundary data in $H^{1,p}(\partial \Omega)$ is solvable (abbreviated $(R)_p$), if for every $f \in H^{1,p}(\partial \Omega) \cap C(\partial \Omega)$ the weak solution $u$ to the problem

$$\begin{cases}
Lu = 0 & \text{in } \Omega \\
u|_{\partial B} = f & \text{on } \partial \Omega
\end{cases}$$

satisfies

$$\|\tilde{N}(\nabla u)\|_{L^p(\partial \Omega)} \lesssim \|\nabla T f\|_{L^p(\partial \Omega)}.$$ 

Again, the implied constant depends only the operator $L$, $p$, and the Lipschitz constant of the domain.
Negative result

Theorem

There exists a bounded measurable matrix $A$ on a unit disk $D$ satisfying the ellipticity condition such that the Dirichlet problem $(D)_p$, the Regularity problem $(R)_p$ and the Neumann problem $(N)_p$ are not solvable for any $p \in (1, \infty)$.

Hence solvability requires extra assumption on the regularity of coefficients of the matrix $A$. 
The Boundary value problems for second order elliptic operators satisfying Carleson condition

Overview of known results

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The Carleson condition - motivation

Consider the boundary value problems associated with a smooth elliptic operator in the region above a graph $t = \varphi(x)$, for $\varphi$ Lipschitz.

Consider a mapping $\Phi : \mathbb{R}_+^n \to \{X = (x, t); t > \varphi(x)\}$ defined by

$$\Phi(X) = (x, c_0 t + (\theta_t * \varphi)(x))$$

where $(\theta_t)_{t>0}$ is smooth compactly supported approximate identity and $c_0$ is large enough so that $\Phi$ is one to one.
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Consider the boundary value problems associated with a smooth elliptic operator in the region above a graph \( t = \varphi(x) \), for \( \varphi \) Lipschitz.

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Then the function \( v = u \circ \Phi \) solves an elliptic equation in \( \mathbb{R}^n_+ \) with coefficients satisfying
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The Carleson condition - motivation

$$\delta(X)^{-1} \left( \text{osc}_{B(x, \delta(x)/2)} a_{ij} \right)^2$$

is the density of a Carleson measure on $\Omega$.

Definition

A measure $\mu$ in $\Omega$ is Carleson if there exists a constant $C = C(r_0)$ such that for all $r \leq r_0$ and $Q \in \partial \Omega$,

$$\mu(B(Q, r) \cap \Omega) \leq C \sigma(B(Q, r) \cap \partial \Omega).$$

The best possible $C$ is the Carleson norm. When $\mu$ is Carleson we write $\mu \in \mathcal{C}$. 
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If \( \lim_{r_0 \to 0} C(r_0) = 0 \), then we say that the measure \( \mu \) satisfies the vanishing Carleson condition, and we write \( \mu \in \mathcal{C}_V \).
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The best possible \( C \) is the Carleson norm. When \( \mu \) is Carleson we write \( \mu \in C \).

If \( \lim_{r_0 \to 0} C(r_0) = 0 \), then we say that the measure \( \mu \) satisfies the vanishing Carleson condition, and we write \( \mu \in C_V \).
Results for Dirichlet problem \((D)_p\)

**Kenig-Pipher, 2001** If

$$\delta(X)^{-1} \left( \text{osc}_{B(X, \delta(X)/2)} a_{ij} \right)^2$$

is a density of Carleson measure on a Lipschitz domain \(\Omega\) then \((D)_p\) is solvable for some (large) \(p < \infty\).

**M.D-Pipher-Petermichl, 2007** For any \(p \in (1, \infty)\) there exists \(C = C(p) > 0\) such that if the Carleson norm bounded is less than \(C(p)\) and the Lipschitz constant \(L\) of the domain \(\Omega\) is smaller than \(C(p)\) then \((D)_p\) is solvable.
Overview of known results

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Results for Neumann and Regularity problems in 2D

M.D.-Rule, 2010 Let $\Omega \subset \mathbb{R}^2$. 
Results for Neumann and Regularity problems in 2D

M.D-Rule, 2010 Let $\Omega \subset \mathbb{R}^2$. For any $p \in (1, \infty)$ there exists $C = C(p) > 0$ such that if the coefficients of $A$ satisfy a variant of our Carleson condition with norm less than $C(p)$ and the Lipschitz constant $L$ of the domain $\Omega$ is smaller than $C(p)$ then $(R)_p$ and $(N)_p$ are solvable.
M.D.-Rule, 2010 Let $\Omega \subset \mathbb{R}^2$.
For any $p \in (1, \infty)$ there exists $C = C(p) > 0$ such that if the coefficients of $A$ satisfy a variant of our Carleson condition with norm less than $C(p)$ and the Lipschitz constant $L$ of the domain $\Omega$ is smaller than $C(p)$ then $(R)_p$ and $(N)_p$ are solvable.
Main theorem-Regularity

Let $1 < p < \infty$ and let $\Omega$ be a Lipschitz domain with Lipschitz norm $L$. Let

$$\delta(X)^{-1} \left( \text{osc}_{B(X,\delta(X)/2)} a_{ij} \right)^2$$

be the density of a Carleson measure on all Carleson boxes of size at most $r_0$ with norm $C(r_0)$. Then there exists $\varepsilon = \varepsilon(\Lambda, n, p) > 0$ such that if $\max\{L, C(r_0)\} < \varepsilon$ then the $(R)_p$ regularity problem

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\begin{cases}
Lu = 0 & \text{in } \Omega \\
u|_{\partial\Omega} = f & \text{on } \partial\Omega \\
\tilde{N}(\nabla u) \in L^p(\partial\Omega)
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is solvable for all $f$ with $\|\nabla^T f\|_{L^p(\partial\Omega)} < \infty$. 
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is solvable for all $f$ with $\|\nabla T f\|_{L^p(\partial\Omega)} < \infty$. Moreover, there exists a constant $C = C(\Lambda, n, a, p) > 0$ such that

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Results for Neumann and Regularity problems in any dimension

Small Carleson norm

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$$(R)_p \iff (D^*)_p'.$$
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Here $\frac{1}{p} + \frac{1}{p'} = 1$ and $(D^*)$ is the Dirichlet problem for the adjoint operator $L^*$. 
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Proof - main ideas

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**The Boundary value problems for second order elliptic operators satisfying Carleson condition**

- **Proof - main ideas**
- **Regularity problem**
Reduction to differentiable coefficients

The idea comes from [DPP]. For a matrix $A$ satisfying our Carleson condition with ellipticity constant $\Lambda$ one can find (by mollifying coefficients of $A$) a new matrix $\tilde{A}$ with same ellipticity constant $\Lambda$ such that $\tilde{A}$ satisfies that
Reduction to differentiable coefficients

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$$\sup\left\{ \delta(X)|\nabla \tilde{A}(Y)|^2; \ Y \in B(X, \delta(X)/2) \right\}$$

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\( p = 2 \) and Square function

Main goal is to establish and two estimates:

\[ \| S(\nabla u) \|_{L^2}^2 \lesssim \text{boundary data} + \varepsilon \| \tilde{N}(\nabla u) \|_{L^2}^2 \]
$p = 2$ and Square function

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$$\|S(\nabla u)\|_{L^2}^2 \approx \|\tilde{N}(\nabla u)\|_{L^2}^2.$$
The Boundary value problems for second order elliptic operators satisfying Carleson condition

Proof - main ideas

Regularity problem

$p = 2$ and Square function

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The boundary value problems for second order elliptic operators satisfying Carleson condition

Proof - main ideas

Regularity problem

The Square function

For any $\nu : \Omega \to \mathbb{R}$ we consider

$$S(\nu)(Q) = \left( \int_{\Gamma(Q)} |\nabla \nu(X)|^2 \delta(X)^{2-n} \, dX \right)^{1/2},$$

for all $Q \in \partial \Omega$. Observe that

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The Square function

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for all \( Q \in \partial\Omega \). Observe that

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First estimate

Key ingredient: Deal separately with

$$\| S(\nabla_T u) \|_{L^2}^2$$
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\[ \| S(\nabla T u) \|_{L^2}^2 \text{ and } \| S(A \nabla u \cdot \nu) \|_{L^2}^2. \]
First estimate

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\[ \| S(\nabla_T u) \|_{L^2}^2 \text{ and } \| S(A \nabla u \cdot \nu) \|_{L^2}^2. \]

We establish a local estimate for \( \| S(\nabla_T u) \|_{L^2}^2 \). In local coordinates we might assume that \( \nabla_T u = (\partial_1 u, \partial_2 u, \ldots, \partial_{n-1} u) \) and \( \nabla_\nu u = \partial_n u. \)
First estimate

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For \( w_k = \partial_k u, \ i = k, 2, \ldots, n-1 \) we use the fact that

\[ \| S(w_k) \|_{L^2}^2 \approx \int_{\mathbb{R}^{n-1}} \frac{a_{ij}}{a_{nn}} \partial_i w_k \partial_j w_k \ t \ dx \ dt. \]
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Next, we integrate by parts. Several type of terms arise, in particular:
First estimate

Key ingredient: Deal separately with

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We establish a local estimate for \( \| S(\nabla_T u) \|_{L^2}^2 \). In local coordinates we might assume that \( \nabla_T u = (\partial_1 u, \partial_2 u, \ldots, \partial_{n-1} u) \) and \( \nabla_{\nu} u = \partial_n u \).

For \( w_k = \partial_k u, \ i = k, 2, \ldots, n-1 \) we use the fact that

\[ \| S(w_k) \|_{L^2}^2 \approx \int_{\mathbb{R}^{n-1}} \frac{a_{ij}}{a_{nn}} \partial_i w_k \partial_j w_k \ t \ dx \ dt. \]

Next, we integrate by parts. Several type of terms arise, in particular:
Terms of the estimate:

\[
\int\int_{\mathbb{R}^n} \frac{1}{a_{nn}} w_k(Lw_k)t \, dxdt
\]

Here we use the fact that for \( k < n \) the PDE \( w_k \) satisfies has relatively “nice” right hand side.
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The Boundary value problems for second order elliptic operators satisfying Carleson condition

Proof - main ideas

Regularity problem

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Difficult direction: \[ \| N(\nabla u) \|_{L^2}^2 \lesssim \| S(\nabla u) \|_{L^2}^2. \]
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Neumann problem - solving for $p = 2$ and induction for integer $p$

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Solve the Neumann problem for $p = 2$ (using solvability of the $(R)_2$).

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The induction step:

Let \( p \geq 2 \) be an integer, \( 0 \leq k \leq p - 2 \) and integer and \( u \) be a solution to \( Lu = \text{div}A \nabla u = 0 \). Then there exists \( \varepsilon > 0 \) such that if the Carleson norm of the coefficients \( C(r_0) < \varepsilon \) then for some \( K = K(\Omega, \Lambda, n, \varepsilon, m, k) > 0 \)

\[
\int\int_{\Omega_r} |\nabla_T u|^{p-k-2} |H|^k |\nabla H|^2 \delta(X) \, dX
\]

\[
\leq (p - k - 2)K \int\int_{\Omega_{2r}} |\nabla_T u|^{p-k-3} |H|^{k+1} |\nabla H|^2 \delta(X) \, dX
\]

\[
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