THE 3D EULER EQUATIONS WITH INFLOW, OUTFLOW AND VORTICITY BOUNDARY CONDITIONS

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Abstract. The 3D incompressible Euler equations in a bounded domain are most often supplemented with impermeable boundary conditions, which constrain the fluid to neither enter nor leave the domain. We establish the well-posedness of solutions with inflow, outflow of velocity when either the full value of the velocity is specified on inflow, or only the normal component is specified along with the vorticity (and an additional constraint). We do this for multiply connected domains and establish compatibility conditions to obtain arbitrarily high H"older regularity.

Statements and Declarations. The authors have no competing interests to declare that are relevant to the content of this article.

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PART 1: OVERVIEW

1. INTRODUCTION

The Euler equations on a bounded, connected domain $\Omega$ in $\mathbb{R}^3$ can be written

$$
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p = f & \text{in } Q, \\
\text{div } u = 0 & \text{in } Q, \\
u(0) = u_0 & \text{on } \Omega,
\end{cases}
$$

where $T > 0$ is fixed and we define the time-space domain,

$$Q := (0, T) \times \Omega.$$  

Here, $u$ is the velocity field of a constant density incompressible fluid, $p$ its scalar pressure, $f$ the divergence-free external force tangential to the boundary, and $u_0$ the initial velocity.

Most often, (1.1) is supplemented with impermeable boundary conditions, $u \cdot n = 0$ on $\Gamma := \partial \Omega$, where $n$ is the outward unit normal vector, meaning that fluid neither enters nor exits the domain. This places one constraint on the velocity field, in sympathy with the equations being first order. Impermeable boundary conditions are not, however, the only possibility. To motivate this work, let us first explore what other options there might be.

Possible boundary conditions. We will argue somewhat heuristically, not attempting to be rigorous, leaving rigor to our proof of well-posedness of the boundary conditions we develop. Some of what we observe will echo observations in [29]—in particular, the comments on an “open boundary” in Section 2 of [29] on the linearized compressible Euler equations and in Section 3 of [29] on the linearized incompressible Euler equations.

By taking the divergence of (1.1)$_1$, the pressure can be recovered from the velocity field by

$$
\begin{cases}
\Delta p = -\nabla u \cdot (\nabla u)^T & \text{in } \Omega, \\
\nabla p \cdot n = \partial_t u \cdot n - (u \cdot \nabla u) \cdot n & \text{on } \Gamma.
\end{cases}
$$

But also, starting from the Gromeka-Lamb form of the Euler equations, one can easily show (see Proposition 3.1) that any $(u, p)$ that satisfies (1.1) must satisfy, on $\Gamma$, the identity,

$$
\begin{align}
\nabla p \cdot n &= \partial_t u \cdot n - (u \cdot \nabla u) \cdot n \\
\omega \cdot (\nabla u) &= -\partial_t u \cdot u^\tau - \nabla_{\Gamma} \left(p + \frac{1}{2} |u|^2\right) + f - \text{curl}_{\Gamma} u^\tau u^\tau, \\
\omega^\tau &= \text{curl}_{\Gamma} u^\tau.
\end{align}
$$

(1.3)

Here, $\omega := \text{curl } u$. The notation we use here is as follows: For any vector field $v$,

$$v^\tau := v \cdot n, \quad v^\perp := v - v^\tau n \text{ on } \Gamma.$$ 

If we impose $u^\tau \equiv 0$ on $\Gamma$ then the vorticity term disappears in (1.3) and there is no constraint on the vorticity. But on portions of the boundary where $u \cdot n$ does not vanish, (1.3) gives a relation among $\omega$, $\nabla_{\Gamma} p$, $u \cdot n$, and $u^\tau$ on the boundary. At the same time, (1.2) gives a (global) relation between $u \cdot n$ (via its time derivative) and $p$. At the risk of oversimplifying, together, (1.2) and (1.3) give two relations among four quantities, so we must have an independent means of determining two of them so as to obtain the value of the other two.

To better understand the consequences of (1.3), we turn to the vorticity equation,

$$
\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = g := \text{curl } f,
$$

(1.4)
obtained by applying curl to both sides of (1.1). This means that the vorticity at time $t$ is the initial vorticity pushed forward (transported and stretched) by the flow. (In 2D, vorticity is simply transported by the flow.)

For impermeable boundary conditions, one can express a Lagrangian solution to the vorticity equation by introducing the flow map, $\eta(t_1, t_2; x)$, for $u$. This flow map gives the position that a particle at $x \in \overline{\Omega}$ at time $t_1$ will be as it moves, forward or backward, along the flow line to time $t_2$. Given the flow map, $\omega(t, x) := \nabla \eta(0, t; \eta(t, 0; x)) \omega_0(\eta(t, 0; x))$ is a Lagrangian solution. (In 2D, it would be $\omega(t, x) := \omega_0(\eta(t, 0; x))$.) This works, because $\eta$ maps any point in $\overline{\Omega}$ to another point in $\overline{\Omega}$, so one can always evaluate $\omega_0(\eta(t, 0; x))$.

At points on the boundary at which $u \cdot n < 0$, however, the flow lines enter the domain, and we must have a way of determining, or generating, the vorticity so that it can be transported into the domain. From (1.3), we have at such inflow points,

$$\begin{align*}
\omega^\tau &:= \frac{1}{u^n} \left[ -\partial_t \mathbf{u}^\tau - \nabla \Gamma \left( p + \frac{1}{2} |\mathbf{u}|^2 \right) + f \right]^1 - \frac{1}{u^n} \text{curl}_\Gamma \mathbf{u}^\tau \mathbf{u}^\tau, \\
\omega^n &:= \text{curl}_\Gamma \mathbf{u}^\tau.
\end{align*}$$

There is, however, another constraint: As vorticity is generated on the boundary and pushed forward into the domain, the resulting vorticity must lie in the range of the curl; that is, the vector field that results must actually itself be the vorticity of some divergence-free vector field. In 2D, this is automatic, because vorticity is simply a scalar field. But in 3D, vorticity is in the range of the curl only if it is divergence-free and has vanishing fluxes across each boundary component.

Taking the divergence of (1.4) leads, after some calculation, to the conclusion that $\partial_t \text{div} \omega + u \cdot \nabla \text{div} \omega = 0$; that is, the divergence of the vorticity is transported by the flow. Since $\text{div} \omega_0 = \text{div} \text{curl} \mathbf{u}_0 = 0$, we need only show that $\text{div} \omega = 0$ at inflow points on the boundary. But another calculation gives that on $\Gamma$,

$$U^n \text{div} \omega = g \cdot n - \partial_t \omega \cdot n - \text{div}_\Gamma [\omega^n \mathbf{u}^\tau - \mathbf{u}^n \omega^\tau],$$

where $\text{div}_\Gamma$ is the divergence operator on the boundary (see Appendix B). This leads to the constraint,

$$\partial_t \omega \cdot n + \text{div}_\Gamma [\omega^n \mathbf{u}^\tau - \mathbf{u}^n \omega^\tau] - g \cdot n = 0 \quad (1.6)$$

at inflow points. These calculations are all formal, but are worked out rigorously in detail in Section 6 of [12].

Insuring that (1.6) holds at inflow points is an issue that must be addressed regardless of the manner in which vorticity is generated on the boundary. (Dealing with the external fluxes vanishing is relatively straightforward, and is also treated in Section 6 of [12].)

Now, without prescribing at least the sign of $u \cdot n$ on the boundary, we would have to determine the regions of inflow dynamically. To avoid this considerable difficulty, we impose Dirichlet conditions for $u \cdot n$ on all of $\Gamma$. It remains to select a second condition that allows the constraints in (1.2) and (1.3) to be met.

In this paper, we use as the second condition the value of $\mathbf{u}^\tau$ at inflow points, so that the full inflow velocity is prescribed—so-called, inflow, outflow boundary conditions:

$$\begin{cases}
\mathbf{u} \cdot n = U^n & \text{on } [0, T] \times \Gamma, \\
\mathbf{u} = \mathbf{U} & \text{on } [0, T] \times \Gamma_+.
\end{cases} \quad (1.7)$$
Then (1.5) serves as a nonlocal boundary condition for $\omega$ in terms of the pressure gradient. It is nonlocal because the pressure at any point on $\overline{\Omega}$ depends upon the velocity in the whole domain via (1.2).

It is not obvious, but specifying the full velocity field at inflow points on the boundary and generating the vorticity at inflow points via (1.5) automatically gives (1.6), as we show in Remark 3.6.

Another possibility is to specify the value of $u \cdot n$ and, at inflow points, the value of the vorticity—a so-called vorticity boundary conditions. This, however, does not lead to the constraint in (1.6) being automatically satisfied; rather, we must impose a restriction on our choice of $\omega$. It is not clear how to do this in greatest generality, but by requiring that the prescribed vorticity be tangential to the inflow boundary, we obtain well-posedness nearly for free from the technology we develop to handle inflow, outflow boundary conditions.

(One could also choose to use an independent relation between $\omega$ and $u^T$ for the second condition. This was done by Chemetov and Antontsev [7] for 2D weak solutions in vorticity form, without uniqueness, for Navier friction boundary conditions.)

Once we have points on the boundary at which $u \cdot n < 0$, we must have other points at which $u \cdot n > 0$ else the fluid could not be incompressible. Hence, we must have $0 = \int_{\Gamma} u \cdot n = \int_{\Omega} \text{div} u$. Reflecting upon (1.5), it would be very difficult to handle $u \cdot n$ vanishing at a point or, even worse, changing sign, especially to obtain classical solutions with higher regularity, which is our intent. To avoid this, each boundary component must have $u \cdot n$ strictly negative (inflow), strictly positive (outflow), or vanish identically. If a component has inflow, then at least one other component must have outflow.

Such boundary conditions—prescribing $u \cdot n$ on $\Gamma$ and $u$ at inflow points—are those studied in Chapter 4 of Antontsev, Kazhikhov, and Monakhov’s [3], an English translation of [2]. (The historical notes on page 13 of [3] point out that the results in Chapter 4 were drawn from [15, 16, 17].)

Throughout, we fix $\alpha \in (0, 1)$.

**Inflow, outflow.** With this discussion of boundary conditions in mind, let us explain in detail our setup, which is that of [3] (though we allow multiply connected domains). We assume that $\Omega$ is a bounded domain in $\mathbb{R}^3$, possibly multiply connected, having a boundary that is at least $C^{2,\alpha}$.

Inflow, outflow, no-penetration conditions are placed on separate portions, $\Gamma_+, \Gamma_-, \Gamma_0$ of $\Gamma$, respectively, each consisting of a finite number of components (with $\Gamma_0 = \emptyset$ or $\Gamma_0 = \Gamma$ allowed—see Section 15). Fixing a function $U^\alpha$ on $[0, T] \times \Gamma$, we assume that

$$U^\alpha = 0 \text{ on } \Gamma_0, \quad U^\alpha < 0 \text{ on } \Gamma_+, \quad U^\alpha > 0 \text{ on } \Gamma_-.$$ 

We will require that $u^\alpha = U^\alpha$, so we must impose the constraint, $\int_{\Gamma_+} U^\alpha = -\int_{\Gamma_-} U^\alpha$.

Next let $h$ be any function on $[0, T] \times \Gamma_+$ having sufficient regularity, as prescribed later. Let $\mathcal{U}$ be a divergence-free vector field on $\overline{Q}$ for which $\mathcal{U} \cdot n = U^\alpha$ on $[0, T] \times \Gamma$ and $\mathcal{U}^T = h$ on $[0, T] \times \Gamma_+$. The choice of $\mathcal{U}$ is not unique, but we can if we wish construct one such $\mathcal{U}$ explicitly as in Section 2 of [10]. We start with a vector field $v$ on $[0, T] \times \Gamma$ chosen so that $v^n = U^\alpha$ on $[0, T] \times \Gamma$, $v^T = h$ on $[0, T] \times \Gamma_+$, and $v^T$ is arbitrarily chosen (but with as much regularity}}
as $\mathbf{h}$ on $[0, T] \times (\Gamma_0 \cup \Gamma_-)$. We then solve the stationary Stokes problem,

$$
\begin{aligned}
\Delta \mathbf{u} + \nabla p &= 0 \quad \text{in } \Omega, \\
\text{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\mathbf{u} &= \mathbf{v} \quad \text{on } \Gamma.
\end{aligned}
$$

(1.8)

Then $\text{curl} \mathbf{U}$ is harmonic, $\text{div} \mathbf{U} = 0$, and $\mathbf{U} = \mathbf{v}$ on $\Gamma$.

The system of equations we study, then, are (1.1) with (1.7):

$$
\begin{aligned}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } Q, \\
\text{div} \mathbf{u} &= 0 \quad \text{in } Q, \\
\mathbf{u}(0) &= \mathbf{u}_0 \quad \text{on } \Omega, \\
\mathbf{u} \cdot \mathbf{n} &= \mathbf{U} \quad \text{on } [0, T] \times \Gamma, \\
\mathbf{u} &= \mathbf{U} \quad \text{on } [0, T] \times \Gamma_+.
\end{aligned}
$$

(1.9)

Because of the way we defined $\mathbf{U}$, we have the equivalence,

$$
(1.9)_5 \longleftrightarrow \mathbf{u}^T = \mathbf{h} \text{ on } [0, T] \times \Gamma_+.
$$

Specifying $\mathbf{U}^n$ and $\mathbf{h}$ as opposed to $\mathbf{U}$ is minimal, in that the value of $\mathbf{U}^T$ on $\Gamma_0 \cup \Gamma_-$ is arbitrary and plays no role in our analysis. But we will continue to use $\mathbf{U}^T$ on $\Gamma_+$ rather than $\mathbf{h}$, as it is more suggestive of its role. Also, treating $\mathbf{U}$ as a background flow, as done in [34, 30, 10], is an instructive and useful point of view.

**Function spaces.** Since we work with divergence-free vector fields satisfying the boundary conditions in (1.9)$_{4,5}$, for any $N \geq 0$ we define the affine hyperplanes of $C^{N+1,\alpha}(\Omega)$ and $C^{N+1,\alpha}(Q)$,

$$
\begin{aligned}
C^{N+1,\alpha}_\sigma(\Omega) &:= \{ \mathbf{u} \in C^{N+1,\alpha}(\Omega) : \text{div} \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = \mathbf{U} \text{ on } \Gamma \}, \\
C^{N+1,\alpha}_\sigma(Q) &:= \{ \mathbf{u} \in C^{N+1,\alpha}(Q) : \text{div} \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = \mathbf{U} \text{ on } [0, T] \times \Gamma \}.
\end{aligned}
$$

(1.10)

Only the normal component of $\mathbf{u}$ is specified on the boundary, so only the boundary condition in (1.9)$_4$ is enforced. We also employ the classical space,

$$
H := \{ \mathbf{u} \in L^2(\Omega)^3 : \text{div} \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma \} = H_0 \oplus H_c,
$$

(1.11)

where

$$
H_c := \{ \mathbf{v} \in H : \text{curl} \mathbf{v} = 0 \}, \quad H_0 := H_c^\perp.
$$

(1.12)

This decomposition is explored in more detail in [12], which we will reference for the few facts that we will need here. For $\mathbf{u} \in H$, $P_H \mathbf{u}$ is termed the *harmonic* component of $\mathbf{u}$.

Much as in [12], we define data regularity and compatibility conditions. We define the boundary values (via $\mathbf{U}$) and the forcing $\mathbf{f}$ for all time on $Q_\infty := [0, \infty) \times \Omega$). We will prove existence only for short time, so the values of $\mathbf{U}$ and $\mathbf{f}$ are only meaningful up to that time.

**Definition 1.1.** We say that the data has regularity $N \geq 0$ if

- $\Gamma$ is $C^{N+2,\alpha}$, $\mathbf{U} \in C^{N+2,\alpha}_\sigma(Q_\infty)$, $\mathbf{f} \in C^{\max\{N,1\},\alpha}(Q_\infty) \cap C([0, \infty); H_0)$;
- $\mathbf{u}_0 \in C^{N+1,\alpha}_\sigma(\Omega)$, $\mathbf{u}_0^T = \mathbf{U}_0^T$ on $\Gamma_+$.

Note that if $\mathbf{U}^n \in C^{N+2,\alpha}([0, \infty) \times \Gamma)$ and $\mathbf{h} \in C^{N+2,\alpha}([0, \infty) \times \Gamma_+)$ then $\mathbf{U}$ given by (1.8) is in $C^{N+2,\alpha}_\sigma(Q_\infty)$.

We assumed one more derivative of regularity for $\mathbf{U}$ than for $\mathbf{u}$. This is for two somewhat related reasons, as explained in Remarks 4.1 and 10.4. The assumption of higher regularity
of the forcing for $N = 0$, though it perhaps could be weakened slightly, plays an important role in [12] in establishing solutions to the linearized equations of Theorem 2.2.

**Compatibility conditions.** We refer to conditions at time zero imposed to obtain the existence of solutions with a given regularity as *compatibility conditions*. Required of the conditions are two primary principles:

1. They depend only upon the initial data and $\mathcal{U}$. This way, the conditions apply equally well to the linear approximation of the solution and to the limiting solution itself.
2. They are compatible with being a solution to (1.9); that is, a solution to (1.9) could, in principle, satisfy them.

The conditions we develop will ensure regularity of the solution for short time. It remains an open question whether a regular solution persists for all time even in 2D.

To define the compatibility conditions for data regularity $N \geq 0$, we first recover a unique mean-zero pressure $p^U$ as the solution to (1.2) with $\partial_t u^n$ replaced by $\partial_t U^n$:

$$
\begin{align*}
\Delta p^U &= -\nabla u \cdot (\nabla u)^T \quad \text{in } \Omega, \\
\nabla p^U \cdot n &= \partial_t U^n - (u \cdot \nabla u) \cdot n \quad \text{on } \Gamma.
\end{align*}
$$

(1.13)

(Note that $f \cdot n = 0$ on the boundary.) We know that if $(u, p)$ solves (1.9)$_{1-4}$ then $\partial_t U^n(0) = \partial_t u^n(0)$ on $\Gamma$, so $p^U_0 := p^U(0) = p(0)$ and

$$
\partial_t u(0) = -u_0 \cdot \nabla u_0 - \nabla p^U_0 + f(0).
$$

(1.14)

We then define the $N$-th compatibility condition to be (making $\text{cond}_{-1}$ vacuous)

$$
\text{cond}_N : \text{cond}_{N-1} \text{ and } \partial_t^{N+1} \mathcal{U}^T|_{t=0} + \partial_t^N \left[ u \cdot \nabla u + \nabla p^U - f \right]^T|_{t=0} = 0 \text{ on } \Gamma_+,
$$

(1.15)

after making the substitutions given by (1.14), applied inductively for $N \geq 1$. Then (1.14) plus induction shows that (1.15) involves only the data regardless of the value of $N$.

For $N = 0$, (1.15) is the compatibility condition in (1.10), (1.11) of Chapter 4 of [3].

We explore compatibility conditions in relation to the generation of vorticity on the boundary in Section 3, for that is how we will relate them to the compatibility conditions required for the linear approximations of solutions to (1.9)$_{1-4}$.

**Main result.** The main result of this paper is Theorem 1.2.

**Theorem 1.2.** Assume that the data has regularity $N$ for some integer $N \geq 0$ as in Definition 1.1 and satisfies $\text{cond}_N$ of (1.15). There is a $T > 0$ such that there exists a solution $(u, p)$ to (1.9) with $(u, \nabla p) \in C^{N+1,\alpha}(Q) \times C^{N,\alpha}(Q)$, which is unique up to an additive constant for the pressure.

**Vorticity boundary condition.** We also consider solutions $(u, p, z)$ to the Euler equations with vorticity boundary conditions, where the value of the vorticity on the inflow boundary is given by a function $\mathcal{H}$ on $[0, T] \times \Gamma_+$:

$$
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p = f + z & \quad \text{in } Q, \\
\text{div } u = 0 & \quad \text{in } Q, \\
u(0) = u_0 & \quad \text{on } \Omega, \\
u \cdot n = U^n & \quad \text{on } [0, T] \times \Gamma, \\
curl u = \mathcal{H} & \quad \text{on } [0, T] \times \Gamma_+.
\end{align*}
$$

(1.16)
Here, \( z \) is an harmonic vector field: it can either be prescribed with the harmonic component of \( u \) obtained as part of the solution or, as in Theorem 1.3, its value can be derived if the harmonic component of \( u \) is prescribed (which allows for a uniqueness result).

**Theorem 1.3.** Fix \( u_c \in C^{N+1,\alpha}(Q)\cap C([0,T]; H_c) \). Assume that the data has regularity \( N \) for some integer \( N \geq 0 \) as in Definition 1.1, though we only require that \( U \in C^{\max\{N+1,2\},\alpha}(Q) \), that \( \text{cond}_N \) holds, and that \( u_c(0) = P_{H_c} P_H u_0 \). Also assume that \( H \in C^{\max\{N,1\},\alpha}([0,T] \times \Gamma_+) \) and

\[
H^n = 0, \quad \text{div}_\Gamma[U^n H^T] + g \cdot n = 0 \quad \text{on} \quad [0,T] \times \Gamma_+.
\]

(1.17)

There is a \( T > 0 \) such that there exists a solution \((u, p, z)\) in \( C^{N+1,\alpha}(Q) \times C^{N,\alpha}(Q) \cap (C^{N+1,\alpha}(Q) \cap C([0,T]; H_c)) \) to (1.9) for which \( P_{H_c} P_H u = u_c \). If \( N \geq 1 \) the solution is unique up to an additive constant for the pressure. In addition, \( z(0) = 0 \).

See Remark 2.3 for an explanation of the condition in (1.17).

**Prior work.** As it is for [12], the primary reference for our work is Chapter 4 of [3], which establishes Theorem 1.2 for \( N = 0 \) and simply connected domains. Our approach is an extension of theirs, worked out for multiply connected domains with higher regularity and de-emphasizing the use of curvilinear coordinates, which makes some of our calculations more transparent.

We mention also the work of Petcu [30], who presents a version of the argument in Chapter 4 of [3] specialized to a 3D channel with a constant \( U \), which simplifies and makes clearer some of the arguments in [3]. Section 1.4 of [22] contains an extensive survey of results, both 2D and 3D, related to the problem we are studying here. We also mention the 2D work of Boyer and Fabrie [4, 5] and the recent works [6, 28].

Vorticity boundary conditions were studied in 2D by Yudovich in [13]. Also see the historical comments in Section 1.4 of [22] concerning partial results in 3D.

We also drew ideas from [20], which proves well-posedness of the 3D Euler equations for impermeable boundary conditions in H"older space (for the equivalent of our \( N = 0 \) regularity).

There are many proofs of well-posedness of the Euler equations taking different approaches. This paper follows in the tradition of McGrath’s [26, 25] and Kato’s [14] in obtaining a solution as a fixed point of an operator, employing Schauder’s fixed point theorem. These works were either for the full plane or for bounded domains in the plane with impermeable boundary conditions.

Finally, we mention that higher regularity solutions for inflow, outflow boundary conditions are needed to complete the proof of existing boundary layer expansions (such as [34, 10] and work in progress of the authors). These works assume that suitable compatibility conditions can be determined from which higher regularity solutions to the Euler equations can be obtained. This need was the original motivation for this work: because of this, in Appendix C we give the explicit form of the compatibility conditions for those works.

**Structure of this paper.** This paper consists of three parts, with two appendices.

In Part 1, following the introduction we have just given, we present in Section 2 the key linear tool we will use from [12]. In Section 3, we explore compatibility conditions as they apply to (1.9) and to the linearized approximations of Section 2. This provides just enough background to allow us to describe, in Section 4, our approach to the proof of Theorem 1.2. We then give the proof of our main result, Theorem 1.2, in Section 5. This proof, however, relies upon three propositions, Propositions 5.1 to 5.3, easily enough stated, but whose proofs occupy the remainder of the paper.
In Part 2, we prepare for the proofs of these propositions by summarizing additional background material from [12] and presenting identities and estimates on the flow map, vorticity generated on the boundary, and pressure. In Part 3, we use results primarily from the second part to prove, first, Proposition 5.1, then leverage it to obtain Proposition 5.2. We also give the proof of Proposition 5.3. In the final two short sections of this part, we describe how Theorem 1.3 follows from a simplification of the estimates obtained in the second part, with a similar observation on Theorem 1.2 when impermeable boundary conditions are imposed on the entire boundary.

We describe the organization of Parts 2 and 3 in more detail at the end of Section 5.

Appendix A contains a number of estimates in Hölder spaces, some very standard, some specific to this paper. In Appendix B we construct a convenient coordinate system in an $\varepsilon$-neighborhood of $\Gamma_+$. We use this system to develop properties of the operators $\nabla_{\Gamma}$, $\text{div}_{\Gamma}$, and $\text{curl}_{\Gamma}$ we use in the body of the paper. This allows us to treat the various calculations on the boundary in a coordinate-free manner, which makes the calculations more transparent. Finally, in Appendix C, we give the form that the compatibility conditions reduce to in the special case in which $U^T \equiv 0$ and $U^m$ is constant along $\Gamma_+$ (as occurs in [34, 10]).

We have structured this paper so as to allow the reader to clearly grasp the overall structure of the proof of Theorem 1.2 without being overwhelmed with the many technical complexities underlying it. It is possible to read only Part 1 and have a good idea of how the proof works. At that level, however, the proof departs little from that for classical impermeable boundary conditions. A more in depth reading would involve at least examining the key pressure estimates in Section 10 and a reading of [12], to understand how the compatibility conditions arise.

On notation. Our notation, while fairly standard, has a few subtleties. If $M$ is a matrix, $M^i_n$ refers to the entry in row $i$ of $M$, column $n$; $v^i$ refers to the $i$th entry in the vector $v$, which we always treat as a column vector for purposes of multiplication. If $M$ and $N$ are the same size matrices then $M \cdot N := M^i_n N^i_n$, where here, as always, we use implicit sum notation. If $u$ and $v$ are vectors then the matrix $u \otimes v$ has components $[u \otimes v]^i_n = u^i v^n$. We define the divergence of a matrix row-by-row, so $\text{div} M$ is the column vector with components $[\text{div} M]^i = \partial_n M^i_n$. Hence, $[\text{div}(u \otimes v)]^i = \text{div}(u \otimes v)^i = \partial_n (u^i v^n)$, where $\partial_n$ is the derivative with respect to the $n$th spatial variable. The notation $\nabla$ means the gradient with respect to the spatial variables only; by $D$ we mean the gradient with respect to all variables, time and space. When applied to the flow map $\eta$, we write $\partial_1 \eta$, $\partial_2 \eta$ to mean the derivative with respect to the first, second time variable. Finally, for vector fields $u$ and $v$, we will interchangeably write $u \cdot \nabla v$ and $\nabla v u$, both of which are a vector whose $i$th component is $u^n \partial_n v^i$.

2. The linearized problem

The linearized Euler equations corresponding to the vorticity form of (1.9) are

$$\begin{cases}
\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = g & \text{in } Q, \\
\omega = H & \text{on } [0, T] \times \Gamma_+, \\
\omega(0) = \omega_0 & \text{on } \Omega.
\end{cases}$$

(2.1)

Here, $H$ is given on $[0, T] \times \Gamma_+$, $\omega_0$ is given on $\Omega$, $u$ and $g$ are given on $Q$, and (2.1) is to be solved for $\omega$. We employ the following three types of solution:
(1) Classical Eulerian or simply classical solution to (2.1), by which we mean that (2.1) holds pointwise.

(2) A weak Eulerian solution:

Definition 2.1. We say that ω is a weak (Eulerian) solution to (2.1) if ω = H on [0, T] × Γ+ pointwise, ω(0) = ω₀ in C⁰,α, and ∂ₜ ω + div(ω ⊗ u) − ω · ∇u = g in D'(Q).

Note that ω has sufficient time and boundary regularity that we do not need to enforce the initial and boundary conditions weakly. Also, ω ⊗ u is a regular distribution, so div(ω ⊗ u) is a distribution even for N = 0.

(3) Lagrangian solutions, defined in terms of the flow map, as in the classical case, but adapted to accommodate the inflow of vorticity from Γ. Because we must first introduce some concepts related to this inflow, we defer this to Definition 8.4.

Setting ω = H on [0, T] × Γ+ in (1.6) yields the constraint,

\[ \partialₜ H^n + \text{div}_T[H^n u^T - U^n H^T] - g \cdot n = 0, \tag{2.2} \]

which is required to obtain a solution to (2.1) in which ω(t) is in the range of the curl.

We define the linear compatibility conditions,

\[ \text{lincond}_0 : \ H(0) = \omega_0 \text{ on } \Gamma+, \quad \text{lincond}_1 : \ \text{lincond}_0 \text{ and } \partialₜ H|_{t=0} + u_0 \cdot \nabla \omega_0 - \omega_0 \cdot \nabla u_0 = g(0) \text{ on } \Gamma+, \tag{2.3} \]

where u₀ := u(0), and, for all N ≥ 2,

\[ \text{lincond}_N : \ \text{lincond}_{N-1} \text{ and } \partialₜ^N H|_{t=0} + \partialₜ^{N-1}[u \cdot \nabla \omega - \omega \cdot \nabla u - g]|_{t=0} = 0 \text{ on } \Gamma+. \tag{2.4} \]

In lincondₙ, we replace \( \partialₜ \omega \) by \( -u \cdot \nabla \omega + \omega \cdot \nabla u + g \) evaluated at time zero for N = 1, and proceed inductively for higher time derivatives.

We define the space

\[ \hat{C}^{N+1,\alpha}_{\sigma}(Q) := \{ u : Q \to \mathbb{R}^3 : \text{div} u = 0, u \cdot n = U^n, \partialₜ^D X \in C^{\alpha}(Q), \]

\[ j + |γ| \leq N + 1, j \leq N \}, \]

endowed with the natural norm based upon its regularity. That is, \( \hat{C}^{N+1,\alpha}_{\sigma}(Q) \) is the same as \( C^{N+1,\alpha}_{\sigma}(Q) \), but with one degree of regularity less in time.

Theorem 2.2 ([12]). Assume that the data has regularity N for some N ≥ 0 and that

- \( g := \text{curl} f \),
- \( u \in \hat{C}^{N+1,\alpha}_{\sigma}(Q) \),
- \( H \in C^{\max(N,1),\alpha}([0, T] \times \Gamma+) \),
- lincondₙ holds,
- \( \omega_0 \) is in the range of the curl,
- (2.2) is satisfied on (0, T] × Γ+.

There exists T > 0 and a solution ω to (2.1) in C⁰,α(Q) for which ω(t) is in the range of the curl for all t ∈ [0, T]. When N ≥ 1, the solution is classical Eulerian and unique. When N = 0, the solution is Lagrangian and is also the unique weak Eulerian solution as in Definition 2.1 for which ω(t) is in the range of the curl for all t ∈ [0, T].

Moreover, there exists a unique \( v \in \hat{C}^{N+1,\alpha}_{\sigma}(Q) \) with \( \text{curl} v = \omega \) and \( v(0) = u_0 \) and a mean-zero pressure field \( \pi \) with \( \nabla \pi \in C^{N,\alpha}(Q) \) for which

\[ \partialₜ v + u \cdot \nabla v - u \cdot (\nabla v)^T + \nabla \pi = f. \tag{2.5} \]
The harmonic component \(v_c\) of \(v\) is given explicitly as
\[
v_c(t) := P_{H_c}u(0) + \int_0^t P_{H_c}f(s) \, ds - \int_0^t P_{H_c}P_{H_c}(\Omega(s)u(s)) \, ds, \tag{2.6}
\]
where the antisymmetric matrix \(\Omega := \nabla K[\omega] - (\nabla K[\omega])^T\). Here, \(K\) is the Biot-Savart operator, as in Section 7.

**Remark 2.3.** As applied to the linearized solution given by Theorem 2.2, the condition in (2.2) is a condition on the data, not on the solution. Applied to the fully nonlinear problem, however, the appearance of \(u^T\) in (2.2) makes (2.2) a condition on the solution. Eliminating the term involving \(u^T\) by requiring that the normal component of the vorticity on inflow vanish gives (1.17), which is a condition on the data.

For \(N = 0\), Theorem 2.2 gives that the weak Eulerian solution is a Lagrangian solution. Though not apparent from its statement, the proof in [12] uses that the velocity field \(v\) is recovered from the Eulerian solution. This is important, since although most of our estimates will be on \(\omega\) using the Lagrangian solution, we will also employ estimates on \(v\), so the two types of solution need to be the same. Without the assumption that \(\omega\) is in the range of the curl, a weak Eulerian solution can be shown to exist, but it may be neither unique nor Lagrangian—see [12] for more details.

### 3. Compatibility conditions: linear and nonlinear

We start with a formula for the generation of vorticity on the inflow boundary:

**Proposition 3.1.** Assume that \((u, p)\) satisfies (1.9) in a classical sense and let \(\omega := \text{curl}\, u\). Then on \([0, T] \times \Gamma\),
\[
u^n\omega^\tau = \left[ -\partial_t u^\tau - \nabla \Gamma \left( p + \frac{1}{2} |u|^2 \right) + f \right]^\perp + \text{curl}_\Gamma u^\tau u^\tau, \quad \omega^n = \text{curl}_\Gamma u^\tau.
\]

**Proof.** As on p. 155 of [3], we start with the Gromeka-Lamb form of the Euler equations,
\[
\partial_t u + \nabla \left( p + \frac{1}{2} |u|^2 \right) - u \times \omega - f = 0 \tag{3.1}
\]

The equivalence of (3.1) and (1.9) follows from the identity,
\[
u \cdot \nabla u = -u \times \omega + \frac{1}{2} \nabla |u|^2, \tag{3.2}
\]
which holds as long as \(\omega = \text{curl}\, u\).

From Lemma B.2
\[
[u \times \omega]^\tau = u^n[\omega^\tau]^\perp - \omega^n[u^\tau]^\perp, \tag{3.3}
\]
so restricting (3.1) to \([0, T] \times \Gamma^+\), we have
\[
\partial_t u^\tau + \nabla \Gamma \left( p + \frac{1}{2} |u|^2 \right) - u^n[\omega^\tau]^\perp + \omega^n[u^\tau]^\perp - f^\tau = 0.
\]

Hence, since \((v^\perp)^\perp = -v\) for any tangent vector \(v\),
\[
u^n\omega^\tau = \left[ -\partial_t u^\tau - \nabla \Gamma \left( p + \frac{1}{2} |u|^2 \right) + f^\tau \right]^\perp + \omega^n u^\tau.
\]

The proof is completed by observing that \(\omega^n = \text{curl}_\Gamma u^\tau\) by (B.4) and (B.9). \(\square\)
We see from Proposition 3.1 that for a solution to (1.9)$_{1-4}$ with $\omega := \text{curl} \, u$, we have
\[
\omega = W[u, p] \text{ on } [0, T] \times \Gamma_+,
\]
where $W[u, p]$ is defined on $[0, T] \times \Gamma_+$ by
\[
W^T[u, p] := \frac{1}{Um} \left[ -\partial_t u^T - \nabla q \left( p + \frac{1}{2} |u|^2 \right) + f^T \right] + \frac{1}{Um} \text{curl}_\Gamma u^T u^T,
\]
\[
W^n[u, p] := \text{curl}_\Gamma u^T.
\]

Now let $u$ be any element of $C^{N+1,\alpha}_\sigma(Q)$. We seek to define a function $H$ on $[0, T] \times \Gamma_+$ as a modification of the expression for $W[u, p]$, in such a way that when the data has regularity $N$, at least the following three properties hold:

1. $H$ at time zero can be defined in terms of the initial data and $U$ only.
2. If $(u, p)$ solves (1.9)$_{1-4}$ and $H = W[u, p]$ on $[0, T] \times \Gamma_+$ then $(u, p)$ satisfies (1.9).
3. $H \in C^{N,\alpha}(0, T] \times \Gamma_+)$ when $u \in C^{N+1,\alpha}_\sigma(Q)$.

For all $N \geq 0$, we define such a function $H$ as done in [3] for $N = 0$. First obtain $q$ from $u$ via
\[
\begin{cases}
\Delta q = -\nabla u \cdot (\nabla u)^T & \text{in } \Omega, \\
\nabla q \cdot n = -\partial_t U^m - N[u] & \text{on } \Gamma.
\end{cases}
\]

Here, $N[u]$ is a modification of $(u \cdot \nabla u) \cdot n$ on $[0, T] \times \Gamma_+$, which we define later in (9.2). The salient point now is that if cond$_N$ holds then $\partial_t^j N[u] = \partial_t^j ((u \cdot \nabla u) \cdot n)$ on $\Gamma_+$ at time zero for all $j \leq N$: see Remark 9.3 and also note (9.4).

Finally, define $H$ on $[0, T] \times \Gamma_+$ by
\[
H^T := \frac{1}{Um} \left[ -\partial_t U^T - \nabla q \left( q + \frac{1}{2} |U|^2 \right) + f^T \right] + \frac{1}{Um} \text{curl}_\Gamma U^T u^T,
\]
\[
H^n := \text{curl}_\Gamma U^T.
\]

To obtain $H$, we replaced all terms in the expression for $W[u, p]$ having a derivative on $u^T$ with $U^T$. If cond$_N$ holds then $\partial_t^j q(0) = \partial_t^j p(0)$, where $p$ is the pressure recovered as it would be for a solution to (1.9)$_{1-4}$: this follows from Remark 9.3.

**Proposition 3.2.** As defined in (3.7) the function $H$ satisfies properties (1) through (3).

**Proof.** That property (1) holds follows from the observations above. We will show property (2) in Proposition 5.3 and property (3) in Proposition 11.1. 

Now, if $(u, p)$ solves (1.9)$_{1-4}$ and $\omega := \text{curl} \, u$, then, of course,
\[
\partial_t \omega(0) = \omega_0 \cdot \nabla u_0 - u_0 \cdot \nabla \omega_0 + g,
\]
\[
\partial_t u(0) = -u_0 \cdot \nabla u_0 - \nabla q_0 + f, \tag{3.8}
\]
where $g := \text{curl} \, f$. Moreover, we have the following:

**Proposition 3.3.** Let $u \in C^{N+1,\alpha}_\sigma(Q)$ and let $\omega$ be the solution to (2.1) given by Theorem 2.2. Also as in Theorem 2.2, let $v \in C^{N+1,\alpha}_\sigma(Q)$ with $\text{curl} \, v = \omega$ and $v(0) = u_0$ solve (2.5). Then
\[
\begin{align*}
\partial_t \text{curl} \, v(0) &= \omega_0 \cdot \nabla u_0 - u_0 \cdot \nabla \omega_0 + g, \\
\partial_t v(0) &= -u_0 \cdot \nabla u_0 - \nabla q_0 + f. \tag{3.9}
\end{align*}
\]
Proof. The identity in (3.9)1 is (2.1)1 at time zero. For (3.9)2, since \( v(0) = u(0) = u_0 \), we have, from (2.5),
\[
\partial_t v = -u_0 \cdot \nabla u_0 + u_0 \cdot (\nabla u_0)^T - \nabla \pi + f = -u_0 \cdot \nabla u_0 + \nabla \left( -\pi + \frac{1}{2} |u_0|^2 \right) + f.
\]
Hence, \( P_H(\partial_t v(0)) = P_H(-u_0 \cdot \nabla u_0 + f) \), so
\[
\partial_t v(0) = -u_0 \cdot \nabla u_0 - \nabla r + f
\]
for some \( r \). Comparing this to (2.5), we see that
\[
\nabla r = \nabla \pi(0) - u_0 \cdot (\nabla v_0)^T = \nabla \pi(0) - u_0 \cdot (\nabla u_0)^T = \nabla \pi(0) - \frac{1}{2} \nabla |u_0|^2.
\]
Taking the divergence of (2.5), we see that at time zero, where \( v = u, \pi \) solves
\[
\begin{cases}
\Delta \pi(0) = -\nabla u_0 \cdot (\nabla u_0)^T + \frac{1}{2} \text{div} \nabla |u_0|^2 & \text{in } \Omega, \\
\nabla \pi(0) \cdot n = -\partial_t u^n - (u \cdot \nabla u) \cdot n + \frac{1}{2} \nabla |u|^2 \cdot n & \text{on } \Gamma.
\end{cases}
\]
It follows, using also that \( N[u] = u_0 \cdot \nabla u_0 \) at time zero, that
\[
\nabla r = \nabla \pi(0) - \frac{1}{2} \nabla |u_0|^2 = \nabla q_0,
\]
giving (3.10).
\[\square\]

Now, we cannot a priori calculate \( \partial_t H \) at time zero since we need the value of the solution \( u \) to do so. But we conclude from Proposition 3.3 that if we formally replace time derivatives of \( \omega(0) \) on \( \Gamma_+ \) by the expressions on the right hand side of (3.8)1, then we can use that value of \( \partial_t H |_{t=0} \) in lincond1 of (2.3) to obtain the linear approximations, and lincond1 will also hold for any solution to (1.9)1-4. Further, it is not hard to see that such replacements can be done inductively for higher time derivatives as well and apply to lincond1.

That is to say, we could directly define the \( N \)-th compatibility condition to be
\[
\text{cond}'_N : \text{cond}'_{N-1} \text{ and } \partial_t^N H |_{t=0} = \partial_t^N \omega |_{t=0} \text{ on } \Gamma_+.
\]
(3.11)
after making the substitutions given by (3.8)1 (and defining \( \text{cond}'_{-1} \) to be vacuous). Expressed this way, the conditions are, in fact, the same as the linear compatibility conditions lincondN of (2.3) and (2.4). Since as part of Definition 1.1 we assume that \( u_0^T = U_0^T \) on \( \Gamma_+ \), for \( N = 0 \), \( \text{cond}'_0 \) could be written \( H^T |_{t=0} = \omega_0^T \).

Similarly, Proposition 3.3 shows that we can define the \( N \)-th compatibility condition in terms of the velocity and pressure, as in (1.15):
\[
\text{cond}_N : \text{cond}_{N-1} \text{ and } \partial_t^{N+1} U^T |_{t=0} + \partial_t^{N} [u \cdot \nabla u + \nabla q - f] |_{t=0} = 0 \text{ on } \Gamma_+,
\]
(3.12)
after making the substitutions given by (3.8)2. This is as done in (1.10, 1.11) of Chapter 4 of AKM for \( N = 0 \). The equivalence of the two types of condition follows from Proposition 3.4.

**Proposition 3.4.** For any \( N \geq 0 \), the compatibility conditions \( \text{cond}_N \) and \( \text{cond}'_N \) in (3.11) and (3.12) are equivalent.

Proof. Suppose that \( \text{cond}_0 \) holds. Then from (3.7), (3.3), and \( \omega_0 = \text{curl} u_0 \), \( \text{cond}'_0 \) holds. Using also (3.2) gives the converse, so \( \text{cond}'_0 \iff \text{cond}_0 \).

For the \( N = 1 \) condition, we know from \( \text{cond}_0 \) that \( [U^m H - U^m \omega]^T |_{t=0} = 0 \). Thus,
\[
\partial_t [U^m H - U^m \omega]^T |_{t=0} = U^m \partial_t [H - \omega]^T |_{t=0} + \partial_t U^m [H - \omega] |_{t=0} = U^m [\partial_t H - \partial_t \omega]^T |_{t=0}.
\]
Since $U^n$ never vanishes on $\Gamma_+$, we have,
\[
\text{cond}'_1 \iff \partial_t H(0) = \partial_t \omega(0) \text{ on } \Gamma_+ \iff \partial_t [U^n H(0) - U^n \omega]_{t=0} = 0 \text{ on } \Gamma_+ \\
\iff \text{cond}_1 .
\]
This argument inducts to all $N \geq 0$. \hfill \Box

Proposition 3.5 shows that our choice of $H$ does, in fact, satisfy (2.2).

**Proposition 3.5.** Let $\omega$ be the solution to (2.1) given by Theorem 2.2 with $H$ given by (3.7). Then (2.2) is satisfied on $(0, T] \times \Gamma_+$.

**Proof.** From (3.7) and using that $\text{curl}_\Gamma U^\tau = H^n$ and that $-\text{div}_\Gamma v^\perp = \text{curl}_\Gamma v$ for a tangent vector field $v$ by (B.9), we have
\[
U^n H^\tau - H^n u^\tau = -\partial_t U^\tau - \nabla_\Gamma \left( q + \frac{1}{2} |U|^2 \right) + f \quad \text{perp}.
\]
Hence,
\[
\text{div}_\Gamma (U^n \omega^\tau - \omega^n u^\tau) = \partial_t H^n - \text{curl}_\Gamma f = \partial_t H^n - g \cdot n,
\]
where we also used (B.11) and that $\text{curl}_\Gamma f = \text{curl} f \cdot n = g \cdot n$. This gives (2.2). \hfill \Box

**Remark 3.6.** Suppose that $(u, p)$ is a solution to (1.9)\textsuperscript{1-4}. Changing every occurrence of $U$ in the proof of Proposition 3.5 to $u$, we see that
\[
\text{div}_\Gamma (U^n \omega^\tau - \omega^n u^\tau) = \partial_t \omega^n - g \cdot n,
\]
which is (1.6). As in the proof of Proposition 3.5, the key is that $\text{curl}_\Gamma u = (\text{curl} u)^n$.

**Generating Compatible Initial Data.** It is easy to obtain examples of initial data satisfying the compatibility conditions: simply choose any $u_0$ and $f$ having sufficient regularity, obtain $p_0$ from $u_0$ via (1.13), then choose $U^\tau(0)$ so that on $\Gamma_+$ we have $U(0) = u_0$ and the values of $\partial_t U^\tau(0), \ldots, \partial_t^{N+1} U^\tau(0)$ are chosen in accordance with the compatibility condition.

It would be much harder, though, to start with $U$ and $f$ and find a suitable $u_0$. See also Appendix C.

**4. The Approach**

Before proceeding with the making of our many estimates, let us outline our approach. Fixing $u_0 \in C_{\sigma}^{N+1, \alpha}(\Omega)$ satisfying $\text{cond}_N$, we define an operator $A$ that takes $u \in C_{\sigma}^{N+1, \alpha}(Q)$ and returns $v \in C_{\sigma}^{N+1, \alpha}(Q)$ with $v(0) = u_0$, defined as follows:

- (0) Replace $u(t)$ by $u(t) - u(0) + u_0$.
- (1) Obtain a pressure field $q$ from $u$ via (3.6).
- (2) Obtain by Theorem 2.2 a solution $\omega \in C^{N, \alpha}(Q)$ to (2.1) with $\omega_0 = \text{curl} u_0$ and $H$ given by (3.7).
- (3) Show that (2.2) is satisfied, so $\omega$ is in the range of the curl and so by Theorem 2.2 there exists a unique velocity field $v \in C_{\sigma}^{N+1, \alpha}(Q)$ and pressure $\pi$ satisfying (2.5).
- (4) Set $Au = v$. 

Remark 4.1. Because we assumed higher regularity of $\mathcal{U}$ over that of $u$, the function $H$ has one more derivative of regularity than $W[u,p]$ of (3.5) in step (2). This higher regularity will persist in the limiting solution, where $H = \mathcal{W}[u,p]$. Such higher regularity is needed for the solving the linearized problem in Theorem 2.2 only for $N = 0$, but it is also needed to handle the pressure estimates for all $N \geq 0$—see Remark 10.4.

**Remark 4.2.** Because of step 0, $A$ is a map from the affine space $\hat{\mathcal{C}}^{N+1,\alpha}(Q)$ to the affine space $\mathcal{C}^{N+1,\alpha}(Q)$, a minor technical point that makes it easier to express the various domains to which we will restrict $A$. Restricting the domain by requiring that $u(0) = u_0$ makes $A$ defined on a convex subset. By intersecting the restricted domain with a ball, we obtain a convex, bounded set, which is compactly embedded in a suitable Banach space, allowing us to apply Schauder’s fixed point theorem in the proof of Theorem 1.2 in Section 5.

It follows from Theorem 2.2 that $A$ maps $\mathcal{C}^{N+1,\alpha}(Q)$ to itself and is bounded, as long as we can obtain sufficient control of the pressure so as to control $H$ in step (3). But $A$, which is nonlinear (because of step (1)), need not be continuous. For this, we introduce some new spaces:

**Definition 4.3.** For a fixed $\beta_1, \beta_2 \in (0,1)$ and any integer $N \geq 0$, let $X_{\beta_1, \beta_2}^N$ be the space of all divergence-free vector fields on $Q$ with normal component given by $U^n$ endowed with the norm,

$$\|u\|_{X_{\beta_1, \beta_2}^N} := \|u\|_{C^N, \beta_1(Q)} + \|\text{curl } u\|_{C^N, \beta_2(Q)}.$$

We will sometimes write $X_{\beta_1, \beta_2}$ for $X_0^0$.

**Remark 4.4.** It will follow from Lemma 7.2 that $X_{\alpha, \alpha}^N = \hat{\mathcal{C}}^{N+1,\alpha}(Q)$.

Fixing $\alpha' \in (\alpha, 1)$, we will show that $A: X_{\alpha', \alpha}^N \to X_{\alpha', \alpha}^N$, $A$ has a fixed set $K$ in $X_{\alpha', \alpha}^N$, and $A$ is continuous in $K$ as a subspace of $X_{\beta, \beta}^N$ for any $\beta < \alpha$. Applying Schauder’s fixed point theorem gives the existence of a fixed point. We will show a posteriori that the full inflow, outflow boundary conditions in (1.9) are satisfied.

Because $X_{\alpha', \alpha}^N \subseteq X_{\alpha, \alpha}^N = \hat{\mathcal{C}}^{N+1,\alpha}(Q)$, we can apply Theorem 2.2 to obtain $Au$. Since $Au$ always lies in $\mathcal{C}^{N+1,\alpha}(Q)$, any fixed point of $A$ will lie in $\mathcal{C}^{N+1,\alpha}(Q)$. In terms of norms, we have the inequalities,

$$\|u\|_{X_{\beta, \beta}^N} \leq \|u\|_{X_{\alpha, \alpha}^N} = \|u\|_{\mathcal{C}^{N+1,\alpha}(Q)} \leq \|u\|_{X_{\alpha', \alpha}^N} \leq \|u\|_{\mathcal{C}^{N+1,\alpha}(Q)},$$

where $0 < \beta < \alpha < \alpha' < 1$. Any velocity recovered from the vorticity as in Section 7 will have one more derivative of regularity in space than does the vorticity, but only the same regularity in time.

In constructing solutions, $X_{\alpha, \alpha}^N = \hat{\mathcal{C}}^{N+1,\alpha}(Q)$ would seem the most natural. Then, once a solution is obtained, the Euler equations themselves easily yield one more derivative in time, giving a solution in $\mathcal{C}^{N+1,\alpha}(Q)$. Indeed, this is how it works for the linearized problem, (2.1).

But there are two difficulties. We need $X_{\alpha', \alpha}^N$ for its extra time regularity to control the pressure estimates, which are non-classical. And though we will be able to obtain a fixed set in $X_{\alpha', \alpha}^N$, $A$ will not be continuous in $X_{\alpha', \alpha}^N$. It will, however, be continuous in $X_{\beta, \beta}^N$: this part is classical, at least for $N = 0$.

When obtaining solutions to the 3D Euler equations in the full space, it is possible to first obtain the equivalent of $N = 0$ regularity solutions, then use a kind of bootstrapping to obtain higher regularity solutions, as in Section 4.4 of [21]. Such an approach does not work in our
setting for two reasons: the need to match the inflow of vorticity from the boundary with that from time zero using compatibility conditions, and the lack of a sufficiently usable Biot-Savart kernel for a 3D bounded domain. (Such a kernel for a 3D domain is a recent development of [8], but even in 2D the required estimates for higher derivatives, simple in the full plane, would make the approach difficult.) In 2D there is an elegant approach of Marchioro and Pulvirenti in [24] (which originates in their earlier text [23]), but it is specific to 2D because it relies upon higher regularity of initial vorticity automatically being propagated by the flow map, whereas that is no longer the case in 3D, where there is stretching.

5. Proof of well-posedness with inflow, outflow

Following the approach laid out in Section 4, we will show that Theorem 1.2 follows from Propositions 5.1 to 5.3, whose technically complex proofs we give in the sections that follow.

Proposition 5.1. Assume that the data has regularity \( N \geq 0 \) and \( u_0 \in C^{N+1,\alpha}_\sigma(\Omega) \). For any \( M > \| u_0 \|_{C^{N+1,\alpha}_\sigma(\Omega)} \) there exists \( T > 0 \) for which the set

\[
K := \{ u \in X^{N}_{\alpha',\alpha} : u(0) = u_0, \| u \|_{X^{N}_{\alpha',\alpha}} \leq M \}
\]

is invariant under \( A \). That is, \( \| u \|_{X^{N}_{\alpha',\alpha}} \leq M \implies \| Au \|_{X^{N}_{\alpha',\alpha}} \leq M \) for any \( u \) in \( K \).

Proof. Given in Section 11. \( \square \)

Proposition 5.2. For any \( \beta \in (0, \alpha) \), \( A : K \to K \) is continuous in the \( X^{N}_{\beta,\beta} \) norm.

Proof. Given in Section 12, and rests strongly upon Proposition 5.1. \( \square \)

Proposition 5.3. Assume that \( (u, p) \in C^{1,\alpha}(Q) \times C^{\alpha}(Q) \) solves \( (1.9)_{1,4} \) and \( \omega := \text{curl} \ u = H \) on \( [0,T] \times \Gamma_+ \), with \( H \) given in (3.7). Then \( (1.9)_5 \) also holds.

Proof. Given in Section 13. \( \square \)

We can now see that Theorem 1.2 is a fairly simple consequence of Propositions 5.2 and 5.3:

Proof of Theorem 1.2. Choose any \( \beta \in (0, \alpha) \). Because \( C^{N,\alpha} \) is compactly embedded in \( C^{N,\beta} \), we see that \( K \) is a convex compact subset of \( X^{N}_{\beta,\beta} \). By Proposition 5.2, \( A \) is continuous as a map from \( K \) to \( K \) in the \( X^{N}_{\beta,\beta} \) norm, and so has a fixed point \( u \) by Schauder’s fixed point theorem. It follows that \( Au = u \) with \( u \in X^{N}_{\alpha',\alpha} \) and hence, in particular, \( u \in C^{N+1,\alpha}_\sigma(Q) \).

Since \( v := Au = u \), it follows from Theorem 2.2 that \( \partial_t u + u \cdot \nabla u + \nabla p = f \) for some pressure \( p \). Hence, \( (u, p) \) is a solution to \( (1.9)_{1,4} \). But since \( u = Au \), we have \( \omega := \text{curl} \ u = H \). Proposition 5.3 thus gives that \( (1.9)_5 \) holds, so \( (u, p) \) is a solution to \( (1.9) \).

To prove uniqueness, let \( (u_1, p_1), (u_2, p_2) \) be two solutions to \( (1.9) \) with the same initial velocity in \( C^{1,\alpha} \) (so we prove uniqueness for \( N = 0 \) and it then follows for all \( N \geq 0 \)). Letting \( w = u_1 - u_2 \), subtracting \( (1.9)_1 \) for \( (u_2, p_2) \) from \( (1.9)_1 \) for \( (u_1, p_1) \),

\[
\partial_t w + u_1 \cdot \nabla w + w \cdot \nabla u_2 + \nabla (p_1 - p_2) = 0.
\]

Multiplying by \( w \) and integrating over \( \Omega \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| w \|^2 = -\int_\Omega (w \cdot \nabla u_2) \cdot w - \frac{1}{2} \int_\Omega u_1 \cdot \nabla |w|^2 \leq \| \nabla u_2 \|_{L^\infty(\Omega)} \| w \|^2 - \frac{1}{2} \int_\Omega u_1 \cdot \nabla |w|^2.
\]

But,

\[
-\int_\Omega u_1 \cdot \nabla |w|^2 = -\int_\Gamma (u_1 \cdot n) |w|^2 = -\int_{\Gamma_-} (u_1 \cdot n) |w|^2 \leq 0,
\]
since \( w = 0 \) on \( \Gamma_+ \), \( u_1 \cdot \mathbf{n} = 0 \) on \( \Gamma_0 \), and \( u_1 \cdot \mathbf{n} > 0 \) on \( \Gamma_- \). Hence,
\[
\frac{d}{dt} \| w \|^2 \leq 2 \| \nabla u_2 \|_{L^\infty(Q)} \| w \|^2,
\]
and we conclude that \( w = 0 \) by Grönwall’s lemma. This gives the uniqueness in Theorem 1.2. \( \square \)

If \( \Gamma_0 = \Gamma \), meaning that classical impermeable boundary conditions are imposed on the entire boundary, then the proof of Theorem 1.2 still applies. Aspects of the proofs of Propositions 5.1 and 5.2 do simplify, however, as we briefly discuss in Section 15.

**Organization of Part 2.** The proof of Proposition 5.1, the first and most technically demanding of the three propositions upon which our proof of Theorem 1.2 relied (indirectly, through Proposition 5.2), will occupy the largest part of the remainder of this paper. To prove it, we summarize in Sections 7 and 8 some of the results from [12], describe the generation of vorticity on the boundary in Section 9, and obtain critical estimates on the pressure in Section 10.

First, however, we introduce in Section 6 some conventions that we will use throughout the remainder of this paper to streamline the presentation.

**Organization of Part 3.** In Section 11 we give the proof of Proposition 5.1 by first obtaining sufficient estimates on the operator \( A \) using (primarily) the pressure estimates from Section 10 along with the estimates on the flow map from Section 8. In Section 12, we use these estimates on \( A \) and the invariant set of Proposition 5.1 to prove Proposition 5.2. In Section 13, we give the proof of Proposition 5.3. In the final two sections of Part 3, we prove Theorem 1.3 and describe the simplifications that occur for fully impermeable boundary conditions.

## Part 2: Preliminary Estimates

### 6. Some conventions

**Pressures and their notation.** Four types of pressures appear in this paper:

- \( p \): The “true” pressure recovered by (1.2), appearing in a solution to (1.9)\textsubscript{1-4}.
- \( q \): The “approximating” pressure recovered by (3.6), used to obtain \( H \) on \([0,T] \times \Gamma_+\).
- \( p^U \): Recovered by (1.13), and used to define the compatibility conditions.
- \( \pi \): The “linearized” pressure of (2.5), obtained by recovering the velocity from the vorticity for the linearized Euler equations.

The true and approximating pressures, \( p \) and \( q \), are the keys, with the majority of our estimates involving \( q \). The pressure \( p^U \) was convenient for most directly defining the compatibility conditions, but since \( \partial_j q(0) = \partial_j p^U(0) \) for all \( j \leq N \), it is not strictly needed. And \( \nabla q = \nabla \pi - (1/2) \nabla |u|^2 \), much like \( r \) and \( \pi \) in (3.10), so \( \pi \) enters only (indirectly) into the proof we gave of Proposition 3.3.

**Constants.** It will simplify notation to write \( M \) as a universal but unspecified bound on \( \| u \|_{X^{N}_{\alpha',\alpha}} \). Thus, we assume that
\[
\| u \|_{X^{N}_{\alpha',\alpha}} \leq M \text{ for some } M \geq 1
\]
in what follows. (Having \( M \geq 1 \) simplifies the form of some estimates.)
Define the solution space for vorticity, any
If Lemma 7.2.

Assume that Lemma 7.1.
and if also the spaces \( H \), We need a few facts from [12] related to the Biot-Savart law, which we present now. We use

\[ C_t \] \( H \) hence, in the final forms of estimates, we will only keep the lowest exponents of \( T \), as does \( U^{-1}_m \): nonetheless, we will think of \( c_0 \) as bounding the size of the initial data.

Remark 6.2. Many of our estimates contain factors of the form \( C_1 T^{e_1} + C_2 T^{e_2} + C_3 T^{e_3} \), \( 0 < e_1 < e_2 < e_3 \), where \( C_1, C_2, \) and \( C_3 \) may depend upon the norms of the data or the solution, but have no explicit dependence on time. To simplify matters, we will assume that \( T \leq T_0 \) for some fixed but arbitrarily large \( T_0 > 0 \). Then

\[
C_1 T^{e_1} + C_2 T^{e_2} + C_3 T^{e_3} \leq C_1 T^{e_1} + C_2 T^{e_2} T_0^{e_2 - e_1} + C_3 T^{e_1} T_0^{e_3 - e_1} \leq C' T^{e_1},
\]

\( C' := (1 + T_0^{e_2 - e_1} + T_0^{e_3 - e_1}) \max\{C_1, C_2, C_3\} \).

Hence, in the final forms of estimates, we will only keep the lowest exponents of \( T \) and, similarly, of \( |t_1 - t_2| \) for \( t_1, t_2 \in [0, T] \).

7. Recovering velocity from vorticity

We need a few facts from [12] related to the Biot-Savart law, which we present now. We use the spaces \( H, H_c, \) and \( H_0 \) of (1.11) and (1.12).

Lemma 7.1. Assume that \( \Gamma \) is \( C^{n,\alpha} \)-regular and let \( X \) be any function space that contains \( C^{n,\alpha}(\Omega)^3 \). For any \( \mathbf{v} \in H \),

\[
\| P_{H_c} \mathbf{v} \|_X \leq C(X) \| \mathbf{v} \|_H
\]

and if also \( \mathbf{v} \in X \), then

\[
\| \mathbf{v} \|_X \leq \| P_{H_c} \mathbf{v} \|_X + C(X) \| \mathbf{v} \|_H, \quad \| P_{H_0} \mathbf{v} \|_X \leq \| \mathbf{v} \|_X + C(X) \| \mathbf{v} \|_H.
\]

For any \( \omega \) in the range of the curl, \( \text{curl}(H^1_0(\Omega)^3) \), there exists a unique \( \mathbf{u} = K[\omega] \in H_0 \cap H^1(\Omega)^3 \) for which \( \text{curl} \mathbf{u} = \omega \). The operator \( K \), which recovers the unique divergence-free vector field in \( H_0 \) having a given curl, encodes the Biot-Savart law.

There exists a vector field \( \mathbf{v} \) as regular as \( \mathbf{U} \) with \( \text{div} \mathbf{v} = 0 \), \( \text{curl} \mathbf{v} = 0 \), and \( \mathbf{v} \cdot \mathbf{n} = U^n \) on \( [0, T] \times \Gamma \). We define

\[
K_{U^n}[\omega] := K[\omega] + \mathbf{v}.
\]

Define the solution space for vorticity, \( V^{N,\alpha}_\sigma(Q) := \{ \omega : C^{N,\alpha}(Q) : \omega(t) \in \text{curl}(H^1(\Omega)^3) \text{ for all } t \in [0, T] \} \).

Lemma 7.2. If \( \Gamma \) is \( C^{N+2} \) then \( K_{U^n} \) maps \( C^{N,\alpha}(\Omega) \cap \text{curl}(H^1(\Omega)^3) \) continuously onto \( C^{N+1,\alpha}_\sigma(Q) \) and \( \text{curl}(H^1(\Omega)^3) \), and maps \( W^{N,p}(\Omega) \cap \text{curl}(H^1(\Omega)^3) \) continuously into \( W^{N+1,p}(\Omega) \) for any \( p \in (1, \infty) \). Also, \( K_{U^n} \) maps \( V^{N,\alpha}_\sigma(Q) \) continuously onto

\[
\hat{C}^{N+1,\alpha}_\sigma(Q) := \{ \mathbf{u} \in \hat{C}^{N+1,\alpha}_\sigma(Q) : \mathbf{u}(t) \in H_0 + \mathbf{v} \text{ for all } t \in [0, T] \}.
\]
The estimates we will need are given in Lemma 7.3.

**Lemma 7.3.** Assume \( \mathcal{U} \in C^{N+1, \alpha}_\sigma(Q) \). Let \( \omega \in C^\alpha(\Omega) \) be a divergence-free vector field on \( \Omega \) having vanishing external fluxes. For any \( u \in H \) there exists \( u_c \in H_c \) such that \( u := Ku_c[\omega] + u_c \), and for all \( t \in [0, T] \),

\[
\|u(t)\|_{W^{N+1,p}(\Omega)} \leq C\|\omega(t)\|_{W^{N,p}(\Omega)} + \|\mathcal{U}(t)\|_{W^{N+1,p}(\Omega)} + \|u_c(t)\|_{W^{N+1,p}(\Omega)},
\]

\[
\|u(t)\|_{C^{N+1, \alpha}(\Omega)} \leq C\|\omega(t)\|_{C^{N, \alpha}(\Omega)} + \|\mathcal{U}(t)\|_{C^{N+1, \alpha}(\Omega)} + \|u_c(t)\|_{C^{N+1, \alpha}(\Omega)},
\]

\[
\|\nabla u(t)\|_{L^p(\Omega)} \leq C_p\|\omega(t)\|_{L^p(\Omega)} + \|\nabla \mathcal{U}(t)\|_{L^p(\Omega)} + \|\nabla u_c(t)\|_{L^p(\Omega)},
\]

\[
\|u(t)\|_{L^p(\Omega)} \leq C_p\|\omega(t)\|_{L^p(\Omega)} + \|\mathcal{U}(t)\|_{L^p(\Omega)} + \|u_c(t)\|_{L^p(\Omega)}
\]

for all \( p \in (1, \infty) \). In each case, the final term can be replaced by \( C\|u\|_H \), and \( u = K[\omega] \) corresponds to \( \mathcal{U} \equiv 0 \) and \( u_c \equiv 0 \).

**Proof.** The first three inequalities follow from Lemma 7.2. The fourth inequality follows from the third and Poincaré’s inequality, since elements of \( H \) have mean zero. Lemma 7.1 allows us to replace each of the final terms by \( C\|u\|_H \).

In Section 11, we will require a bound on \( \|u\|_{C^{N+1}(Q)} \) that is better than just \( M \) of (6.1). To obtain such a bound, first observe that

\[
\|\omega\|_{L^2(Q)} \leq \left( \int_0^T M^2 \right)^{\frac{1}{2}} \leq MT^{\frac{1}{2}}.
\]

In analogy with \( \dot{C}_{\sigma}^{N+1}(Q) \), we define \( \dot{C}_{\sigma}^{N+1}(Q) \) to be the space \( \dot{C}_{\sigma}^{N+1}(Q) \), but with one fewer derivative of time regularity, and similarly for \( \dot{C}^{N+1}(Q) \). Then, using Lemmas 7.3 and A.5, for any \( 0 < \beta < \alpha \),

\[
\|u\|_{\dot{C}^{N+1}(Q)} \leq \|\mathcal{V}\|_{\dot{C}^{N+1}(Q)} + \|u - \mathcal{V}\|_{\dot{C}^{N+1}(Q)} \leq c_0 + \|u - \mathcal{V}\|_{\dot{C}^{N+1}(Q)}
\]

\[
\leq c_0 + C_\beta\|\omega\|_{C^{N, \beta}} + c_0 + C_\beta\|\omega\|_{L^\infty(Q)} + C_\beta\|\omega\|_{L^\infty(\Omega)}^{a_\alpha} \|\omega\|_{L^2(Q)}^{1 - a_\alpha}
\]

\[
\leq c_0 + C_\beta\|\omega\|_{L^\infty(Q)} + C_\beta MT^{\beta},
\]

where \( 0 < b < 1 \) (its exact value being unimportant). Here, we used our assumptions that \( M \geq 1 \) and \( T \leq T_0 \) to simplify the form of the estimates coming from Lemma A.5 (see Remark 6.2).

**8. Flow map estimates**

The pushforward of the initial vorticity by the flow map meets, along a hypersurface \( S \) in \( Q \), the pushforward of the vorticity generated on the inflow boundary. This requires some analysis at the level of the flow map. For the most part, the analysis in [12], which we summarize here, suffices. The coarse bounds developed on the flow map in [12], however, would only be sufficient for us to obtain small data existence of solutions: for the short time result for general data that we desire, we will require more explicit and refined bounds, which we develop in Lemma 8.2.

We assume throughout this section that \( \mathcal{U} \in C^{N+2, \alpha}_\sigma(Q) \), \( u \in \dot{C}^{N+1, \alpha}_\sigma(Q) \) for some \( N \geq 0 \). As in [12], we extend \( u \) to be defined on all of \( \mathbb{R} \times \mathbb{R}^3 \) using an extension operator like that in Theorem 5, chapter VI of [31]. This extension need not be divergence-free. This extension is really for a matter of convenience in stating results; it is only the value of \( u \) on \( \partial\Omega \) that ultimately concerns us.
We define \( \eta: \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \) to be the unique flow map for \( u \), so that \( \partial_{t_2} \eta(t_1, t_2; x) = u(t_2, \eta(t_1, t_2; x)) \). Then \( \eta(t_1, t_2; x) \) is the position that a particle starting at time \( t_1 \) at position \( x \in \mathbb{R}^3 \) will be at time \( t_2 \) as it moves under the action of the velocity field \( u \).

For any \((t, x) \in \overline{Q}\) let

- \( \gamma(t, x) \) be the point on \( \Gamma_+ \) at which the flow line through \((t, x)\) intersects with \( \Gamma_+ \);
- let \( \tau(t, x) \) be the time at which that intersection occurs.

For our purposes, we can leave \( \tau \) and \( \gamma \) undefined if the flow line never intersects with \( \Gamma_+ \).

**Remark 8.1.** *We will often drop the \((t, x)\) arguments on \( \tau \) and \( \gamma \) for brevity.*

We define the hypersurface,

\[
S := \{(t, x) \in \overline{Q} : \tau(t, x) = 0\}
\]

and the open sets \( U_\pm \subseteq Q \),

\[
U_- := \{(t, x) \in Q : (t, x) \notin \text{domain of } \tau, \gamma\},
\]

\[
U_+ := \{(t, x) \in Q : \tau(t, x) > 0\}.
\]

Then \( S \) is \( C^{N+1, \alpha} \) as a hypersurface in \( Q \) and \( S(t) := \{x \in \Omega : (t, x) \in S\} \) is \( C^{N+1, \alpha} \) as a surface in \( \Omega \).

The estimates on the flow map in Lemma 8.2 are more explicit than in [12], where we required only coarse estimates. We note that \( \eta \) has one more derivative of time regularity (in each time variable) than has \( u \), which we can see in the explicit estimates. In Lemma 8.2, \( \dot{C}^\alpha(Q) \) is the homogeneous Hölder norm and the subscripts \( x \) and \( t \) refer to norms only in those variables (see (A.3) for detailed definitions).

**Lemma 8.2.** *The flow map \( \eta \in C^{N+1, \alpha}([0, T]^2 \times \mathbb{R}^3) \). Define \( \mu: U_+ \to [0, T] \times \Gamma_+ \) by*

\[
\mu(t, x) = (\tau(t, x), \gamma(t, x)).
\]

*The functions \( \tau, \gamma, \mu \) lie in \( C^{N+1, \alpha}(\overline{U_+} \setminus \{0\} \times \Gamma_+) \). Moreover,*

\[
\begin{align*}
\|\partial_t \eta(t_1, t_2; x)\|_{L^\infty} &\leq \|u\|_{L^\infty(Q)} h(t_1, t_2), \\
\|\nabla \eta(t_1, t_2; x)\|_{L^\infty} &\leq h(t_1, t_2), \\
\|\nabla \eta(0, t_2; x)\|_{\dot{C}^\alpha_2(Q)} &\leq \|\nabla u\|_{L^\infty(Q)} h(0, T) T^{1-\alpha}, \\
\|\nabla \eta(0, T; x)\|_{\dot{C}^\alpha(Q)} &\leq e^{(1+2\alpha)MT} MT^{1-\alpha},
\end{align*}
\]

*where*

\[
h(t_1, t_2) := \exp \left[ \int_{t_1}^{t_2} \|\nabla u(s)\|_{L^\infty} \, ds \right] \leq e^{MT}.
\]

*Also,*

\[
\|D\mu\|_{L^\infty(Q)} \leq CU_{\text{min}}^{-1} [1 + \|u\|_{L^\infty(Q)}^2] h(0, T),
\]

*where \( U_{\text{min}} \) is as in (6.2).*
More generally, for any \( N \geq 0 \), defining \( \exp^n \) to be \( \exp \) composed with itself \( n \) times,

\[
\| \partial_t^N \eta(t_1, t_2; x) \|_{L^\infty([0,T]^2 \times \Omega)} \leq C\| u \|_{C^N(Q)} \exp^{N+1}(MT),
\]

\[
\| \nabla^{N+1} \eta(0, t_2; x) \|_{C_{t_2}^N(Q)} \leq \| \nabla^{N+1} u \|_{L^\infty(Q)} \exp^{N+1}(MT) T^{1-\alpha},
\]

\[
\| \nabla^{N+1} \eta(0, t_2; x) \|_{C_{t_2}^N(Q)} \leq \exp^{N+1}(CMT) \int_0^T \| \nabla^{N+1} u(s) \|_{C^\alpha} ds,
\]

\[
\| D_{N+1}^t \mu \|_{L^\infty(Q)} \leq c_0[1 + \| u \|_{C^N(Q)}^{2(N+1)}] \exp^{N+1}(MT).
\]

Proving (8.3).

Proof. We will apply Lemma A.2 multiple times without explicit reference. Taking the gradient of the integral expression in (3.2) of [12],

\[
\nabla \eta(t_1, t_2; x) = I + \int_{t_1}^{t_2} \nabla u(s, \eta(t_1, s; x)) \nabla \eta(t_1, s; x) ds.
\]

Thus,

\[
\| \nabla \eta(t_1, t_2; x) \|_{L^\infty_{t_2}} \leq 1 + \int_{t_1}^{t_2} \| \nabla u(s) \|_{L^\infty_{t_1}} \| \nabla \eta(t_1, s; x) \|_{L^\infty_{t_2}} ds.
\]

Grönwall’s lemma, applied with fixed \( t_1 \), gives (8.1) \( 2 \). Lemma 3.1 of [12] gives \( \partial_t \eta(t_1, t_1; x) = -u(t_1, x) \cdot \nabla \eta(t_1, t_2; x) \), from which (8.1) \( 1 \) follows.

It also follows from (8.4) that

\[
\| \nabla \eta(0, t_2; x) \|_{C_{t_2}^N} \leq \sup_{t_2 \neq t_2'} \frac{\| \nabla u \|_{L^\infty(Q)} \| \nabla \eta \|_{L^\infty(Q)} |t_2 - t_2'|^{1/\gamma}}{\| \nabla \eta \|_{L^\infty(Q)} h(0, T) T^{1-\alpha}},
\]

giving (8.1) \( 3 \).

Returning once more to (8.4),

\[
\| \nabla \eta(t_1, t_2; x) \|_{C_{t_2}^N} \leq \int_{t_1}^{t_2} \| \nabla u(s, \eta(t_1, s; x)) \|_{C_{t_2}^N} ds.
\]

But, using Lemma A.1,

\[
\| \nabla u(s, \eta(t_1, s; x)) \|_{C_{t_2}^N} \leq \| \nabla u(s, \eta(t_1, s; x)) \|_{C_{t_2}^N} \| \nabla \eta(t_1, s; x) \|_{L^\infty_{t_2}} + \| \nabla u(s, \eta(t_1, s; x)) \|_{L^\infty_{t_2}} \| \nabla \eta(t_1, s; x) \|_{C_{t_2}^N}
\]

\[
\leq \| \nabla u(s) \|_{C_{t_2}^N} \| \eta(t_1, s; x) \|_{L^\infty_{t_2}} \| \nabla \eta(t_1, s; x) \|_{C_{t_2}^N} + \| \nabla u(s) \|_{L^\infty_{t_2}} \| \nabla \eta(t_1, s; x) \|_{C_{t_2}^N} \leq \| \nabla u(s) \|_{C_{t_2}^N} h(t_1, s) \| \nabla \eta(t_1, s; x) \|_{C_{t_2}^N},
\]

so

\[
\| \nabla \eta(0, t_2; x) \|_{C_{t_2}^N} \leq \int_0^{t_2} \| \nabla u(s) \|_{C_{t_2}^N} \| \nabla \eta(0, s; x) \|_{C_{t_2}^N} ds + \int_0^{t_2} \| \nabla u(s) \|_{L^\infty_{t_2}} \| \nabla \eta(0, s; x) \|_{C_{t_2}^N} ds.
\]
Applying Grönwall’s lemma gives
\[
\|\nabla \eta(0, t_2; \mathbf{x})\|_{C^\alpha} \leq \left[ h(0, t_2)^{2\alpha} \int_0^{t_2} \|\nabla \mathbf{u}(s)\|_{C^\alpha} \, ds \right] \exp \int_0^{t_2} \|\nabla \mathbf{u}(s)\|_{L^\infty(\Omega)} \, ds,
\]
which is (8.1).4

From Lemma 3.5 of [12],
\[
\partial_t \tau = -U^n(\tau, \gamma)^{-1} \partial_t \eta(t, \tau; \mathbf{x}) \cdot \mathbf{n}(\gamma), \quad \nabla \tau = -U^n(\tau, \gamma)^{-1} (\nabla \eta(t, \tau; \mathbf{x}))^T \mathbf{n}(\gamma),
\]
\[
\partial_t \gamma = \partial_t \eta(t, \tau; \mathbf{x}) + \partial_t \tau \mathbf{u}(t, \gamma), \quad \nabla \gamma = \mathbf{u}(t, \gamma) \otimes \nabla \tau + \nabla \eta(t, \tau; \mathbf{x}).
\]

We use these expressions to calculate,
\[
\|\partial_t \tau\|_{L^\infty(Q)} \leq CU_{\text{min}}^{-1} \|\partial_t \eta\|_{L^\infty(Q)} \leq CU_{\text{min}}^{-1} \|\mathbf{u}\|_{L^\infty(Q)} h(0, T),
\]
\[
\|\nabla \tau\|_{L^\infty(Q)} \leq CU_{\text{min}}^{-1} \|\nabla \eta\|_{L^\infty(Q)} \leq CU_{\text{min}}^{-1} \|\mathbf{u}\|_{L^\infty(Q)} h(0, T),
\]
\[
\|\partial_t \gamma\|_{L^\infty(Q)} \leq CU_{\text{min}}^{-1} \|\partial_t \eta\|_{L^\infty(Q)} + \|\mathbf{u}\|_{L^\infty(Q)} \|\partial_t \tau\|_{L^\infty(Q)}
\leq CU_{\text{min}}^{-1} \|\mathbf{u}\|_{L^\infty(Q)} + \|\mathbf{u}\|_{L^2(Q)}^2 h(0, T),
\]
\[
\|\nabla \gamma\|_{L^\infty(Q)} \leq \|\mathbf{u}\|_{L^\infty(Q)} \|\nabla \tau\|_{L^\infty(Q)} + \|\nabla \eta\|_{L^\infty(Q)} \leq [1 + CU_{\text{min}}^{-1} \|\mathbf{u}\|_{L^\infty(Q)}] h(0, T).
\]

Summing these bounds gives the bound on \(D\mu = (\partial_t \mu, \nabla \mu).

The bounds for higher \(N\) follow from inductive extension of these arguments. \(\square\)

**Remark 8.3.** The exact bounds in Lemma 8.2 are not so important, but it is important that \(M\) only appear in the exponentials, while multiplicative factors be for lower norms of \(\mathbf{u}\) than \(X^N\), as these can be a little better bounded (by (7.2), primarily).

We are now in a position to give the definition of a Lagrangian solution to (2.1), as it appears in [12]. For this purpose, define
\[
\gamma_0 = \gamma_0(t, \mathbf{x}) := \eta(t, 0; \mathbf{x}).
\]
As with \(\tau\) and \(\gamma\) (see Remark 8.1) we will often drop the \((t, \mathbf{x})\) arguments on \(\gamma_0\).

**Definition 8.4** (Lagrangian solution to (2.1)). Define \(\omega_{\pm}\) and \(G_{\pm}\) on \(U_{\pm}\) by
\[
\omega_-(t, \mathbf{x}) = \nabla \eta(0, t; \gamma_0) \omega_0(\gamma_0) + G_+(t, \mathbf{x}),
\]
\[
\omega_+(t, \mathbf{x}) = \nabla \eta(\tau, t; \gamma) \mathbf{H}(\tau, \gamma) + G_-(t, \mathbf{x}),
\]
\[
G_-(t, \mathbf{x}) := \int_0^t \nabla \eta(s, t; \eta(t, s; \mathbf{x})) \mathbf{g}(s, \eta(t, s; \mathbf{x})) \, ds, \quad (8.6)
\]
\[
G_+(t, \mathbf{x}) := \int_{\tau(t, \mathbf{x})}^t \nabla \eta(s, t; \eta(t, s; \mathbf{x})) \mathbf{g}(s, \eta(t, s; \mathbf{x})) \, ds.
\]

Then \(\omega\) defined by \(\omega|_{U_{\pm}} = \omega_{\pm}\) is called a Lagrangian solution to (2.1).

In (8.6), we left the value of \(\omega\) along \(S\) unspecified. Under the assumptions of Theorem 2.2, \(\omega_{\pm}\) can be extended along \(S\) so that \(\omega\) lies in \(C^{N,\alpha}(Q)\), and the bounds on \(U_{\pm}\) combine to give estimates on \(\omega\) in \(C^{N,\alpha}(Q)\).
9. The nonlinear term on the boundary

Proposition 9.2 gives coordinate-free expressions for \((u \cdot \nabla u) \cdot n\). The proof of Proposition 9.2 is most readily obtained using the boundary coordinates introduced in Appendix B, so we defer it to that appendix.

**Definition 9.1.** For any tangent vector field \(v\) on \(\Gamma\), define \(v^\perp\) to be \(v\) rotated 90 degrees counterclockwise around the normal vector when viewed from outside \(\Omega\) (so \(v^\perp = n \times v\)).

We will write \(A\) for the shape operator on the boundary, so for any tangential vector field, \(Av\) is the directional derivative of \(n\) in the direction of \(v\), which we note is also a tangential vector field. We write the gradient and divergence on the boundary as \(\nabla_\Gamma\) and \(\text{div}_\Gamma\), giving a fuller definition in Appendix B.

**Proposition 9.2.** Assume that \(\Gamma\) is \(C^2\). Let \(u\) be a divergence-free differentiable vector field, let \(u^n = u \cdot n\), and let \(u^\tau = u - u^n n\), treating it as a vector in the tangent bundle to \(\Gamma\). Let \(\kappa_1, \kappa_2\) be the principal curvatures on \(\Gamma\). On \([0,T] \times \Gamma\), we have

\[
(u \cdot \nabla u) \cdot n = u^\tau \cdot \nabla_\Gamma u^n - u^n \text{div}_\Gamma u^\tau - (\kappa_1 + \kappa_2) (u^n)^2 - u^\tau \cdot Au^\tau.
\]

(9.1)

If \(u = \mathcal{U}\) on \(\Gamma_+\) then \((u \cdot \nabla u) \cdot n = (\mathcal{U} \cdot \nabla \mathcal{U}) \cdot n\) on \(\Gamma_+\).

The nonlinear term on the boundary is key to recovering the pressure, as we will see in the next section. For these purposes, we define

\[
N[u] := \begin{cases} 
(u \cdot \nabla u) \cdot n & \text{on } [0,T] \times (\Gamma_- \cup \Gamma_0), \\
(u \cdot \nabla u) \cdot n + \text{div}_\Gamma (U^n(u^\tau - \mathcal{U}^\tau)) & \text{on } [0,T] \times \Gamma_+.
\end{cases}
\]

(9.2)

Using that \(u^n = \mathcal{U}^n\), substituting the expression in (9.1) for \((u \cdot \nabla u) \cdot n\), and using (B.8), we see that on \(\Gamma_+\),

\[
N[u] = 2u^\tau \cdot \nabla_\Gamma U^n - \text{div}_\Gamma (U^n(u^\tau - \mathcal{U}^\tau)) - (\kappa_1 + \kappa_2) (U^n)^2 - u^\tau \cdot Au^\tau,
\]

(9.3)

so \(N[u]\) has no derivatives on \(u^\tau\). Nonetheless, integrating (9.2) by parts along each boundary component using Lemma B.1, we see that

\[
\int_\Gamma N[u] = \int_\Gamma (u \cdot \nabla u) \cdot n,
\]

(9.4)

which will allow us to use \(N[u]\) in place of \((u \cdot \nabla u) \cdot n\) in the Neumann boundary condition on the pressure in Section 10.

**Remark 9.3.** If \(\text{cond}_N\) holds, then \(\partial_t^j N[u] = \partial_t^j ((u \cdot \nabla u) \cdot n)\) on \(\Gamma_+\) at time zero for all \(j \leq N\), and once we obtain a solution to (1.9), we will have \(N[u(t)] = (u(t) \cdot \nabla u(t)) \cdot n\) for all \(t \in [0,T]\). Moreover, Proposition 5.3 shows that (1.9) is satisfied for the solution as well.

**Remark 9.4.** We used the explicit expression in Proposition 9.2 only to show that \(N[u]\) has no derivatives on \(u^\tau\) on the inflow boundary, and hence has one more tangential derivative of regularity there than we might expect.
10. Pressure Estimates

We can determine the pressure from the velocity by taking the divergence of (1.9)\textsubscript{1} and using that \(\text{div } u = 0\), which yields (1.2). On \(\Gamma_0\), as we can see from (9.1), \(\nabla p \cdot n = -u^T \cdot A u^T\) (= \(-\kappa |u|^2\) in 2D). Hence, when \(\Gamma = \Gamma_0\), standard Schauder estimates imply that \(\nabla p\) and \(u\) have the same spatial regularity. This is the classical impermeable boundary case. But for inflow, outflow boundary conditions, the expression for \(\nabla p \cdot n\) contains spatial derivatives of \(u\), as we can see from (9.1), and elliptic theory gives only a pressure gradient of one fewer spatial derivative than the velocity. (Because \(u \cdot n = U^n\) on all of \(\Gamma\), the time derivative in (1.2)\textsubscript{2} does not impact the regularity of \(p\).

We see, then, that classical impermeable boundary conditions are very special, and with inflow, outflow we should not expect to obtain a gradient pressure field of spatial regularity equal to that of \(u\). This is not in itself a problem, for as we can see from (3.7), we only need the pressure gradient to have the same regularity as the vorticity to generate vorticity on the boundary. We will need higher regularity, however, to obtain a fixed point for the operator \(A: X_{\alpha',\alpha} \to X_{\alpha',\alpha}\).

We circumvent this difficulty using the simple but clever technique in [3]: we replace the boundary condition in (1.2)\textsubscript{2} using \(N[u]\) of (9.2), solving instead, (3.6) for the pressure \(q\). We see from (9.4) that the required compatibility condition coming from \(\int_{\Gamma} \nabla q \cdot n = \int_{\Omega} \Delta q = \int_{\Omega} \text{div}(-\partial_t u - u \cdot \nabla u)\) remains satisfied when using \(-\partial_t U^n - N[u]\) in place of \(-\partial_t u^n - (u \cdot \nabla u) \cdot n\) on \(\Gamma\).

\begin{lemma}
Suppose that \(\Omega'\) is a compact subset of \(\Omega \cup \Gamma_+, \Omega' \neq \emptyset\). For any \(n \geq 0\),
\[
\|f\|_{W^{n+2,r}((\Omega'))} \leq C \left[ \|\Delta f\|_{W^{n,r}((\Omega))} + \|\nabla f \cdot n\|_{W^{n+1,\frac{1}{2},r}(\Gamma_+)} + \|f\|_{L^r((\Omega))} \right]
\]
(10.1)
for \(f \in W^{n,r}(\Omega)\), where \(r \in (1, \infty)\).
\end{lemma}

\begin{proof}
These bounds for \(n = 0\) are stated near the bottom of page 174 of [3], but let us say a few words about them. First, they are derived from combining an interior estimate away from all boundaries with an estimate that includes only \(\Gamma_+\). Second, [3] does the \(N = 0\) case, and we use (15.1.5) of [1] for the \(N \geq 1\) case.
\end{proof}

We start in Propositions 10.2 and 10.3 by controlling only the spatial derivatives of \(q_i\).

\begin{proposition}
Let \(r \in [2, \infty)\), \(t_1, t_2 \in [0, T]\), and \(q\) solve (3.6) for some \(u \in X_{\alpha',\alpha}\) with \(q\) normalized so that
\[
\int_{\Omega} q|q|^{r-2} = 0.
\]
(10.2)
Then
\[
\|q(t_1)\|_{L^r(\Omega)} \leq C_1, \quad \|q(t_1) - q(t_2)\|_{L^r(\Omega)} \leq C_2 \|u\|_{X_{\alpha',\alpha}} |t_1 - t_2|^{\alpha'},
\]
(10.3)
where
\[
C_1 := C \left[ \|u\|_{X_{\alpha',\alpha}}^2 + \|u\|_{L^\infty(Q)}^2 \right], \quad C_2 := C \left[ \|u\|_{L^\infty(Q)} + \|u\|_{L^\infty(Q)} \right],
\]
the constant \(C\) depending only upon \(\Omega\) and \(r\).
\end{proposition}

\begin{proof}
We adapt the argument on pages 175-176 of [3]. For now we suppress the time variable.
Let $\beta$ be the unique mean-zero solution to
\[
\begin{aligned}
\Delta \beta &= q|q|^{r-2} \quad \text{in } \Omega, \\
\nabla \beta \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma,
\end{aligned}
\]
where the normalization of $q$ in (10.2) permits the existence of the solution $\beta$. Letting $r' = r/(r-1)$, which we note is Hölder conjugate to $r$, Lemma 10.1 gives
\[
\|\beta\|_{W^{2,r'}(\Omega)} \leq C\|q\|^{r-1}_{L^r(\Omega)} = C\|q\|^{r-1}_{L^r(\Omega)}.
\]
Then,
\[
\|q\|_{L^r(\Omega)} = (\Delta \beta, q) = -\nabla \beta, \nabla q) + \int_{\Gamma} (\nabla \beta \cdot \mathbf{n}) q = (\Delta q, \beta) - \int_{\Gamma} (\nabla q \cdot \mathbf{n}) \beta.
\]

Now,
\[
(\Delta q, \beta) = -(\text{div}(\mathbf{u} \cdot \nabla \mathbf{u}), \beta) = (\mathbf{u} \cdot \nabla \mathbf{u}, \nabla \beta) - \int_{\Gamma} U^n(\mathbf{u} \cdot \nabla \beta) - (\mathbf{u} \otimes \mathbf{u}, \nabla \nabla \beta)
\]
and, using (9.2)
\[
-\int_{\Gamma} ((\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{n}) \beta = \int_{\Gamma} (\partial_t U^n + \nabla p \cdot \mathbf{n}) \beta + \int_{\Gamma} \text{div}_\Gamma(\mathbf{u} \otimes \mathbf{u}^T - \mathbf{u}^T - \mathbf{u} \otimes \mathbf{u})
\]
\[
= \int_{\Gamma} (\partial_t U^n + \nabla p \cdot \mathbf{n}) \beta,
\]
the integral over $\Gamma_+$ vanishing by Lemma B.1. Hence,
\[
\|q\|_{L^r(\Omega)} = -\nabla \beta, \nabla \beta) + \int_{\Gamma} (U^n(\mathbf{u} \cdot \nabla \beta) - (\mathbf{u} \otimes \mathbf{u}, \nabla \nabla \beta)). \quad (10.4)
\]
We thus have the bound,
\[
\|q\|_{L^r(\Omega)} \leq \|\mathbf{u}\|_{L^\infty} \|\mathbf{u}\|_{L^r(\Omega)}^{\beta} \|\nabla \beta\|_{L^r(\Gamma)}^{\Omega} + \|\partial_t U^n\|_{L^r(\Gamma)}^{\Omega} \|\nabla \beta\|_{L^r(\Gamma)}^{\Omega} - \int_{\Gamma} U^n(\mathbf{u} \cdot \nabla \beta).
\]
But,
\[
-\int_{\Gamma} U^n(\mathbf{u} \cdot \nabla \beta) \leq \|U\|_{L^r([0,T] \times \Gamma)} \|\mathbf{u}\|_{L^\infty(\Omega)} \|\nabla \beta\|_{L^r(\Gamma)} \quad (10.5)
\]
We see, then, that
\[
\|q\|_{L^r(\Omega)} \leq C_1 \|\beta\|_{W^{2,r'}} \leq C_1 \|q\|^{r-1}_{L^r(\Omega)},
\]
from which (10.3) follows.
To obtain (10.3) we argue the same way, bounding now $q := q(t_1) - q(t_2)$ and using $\partial_t U^n(t_1) - N[u(t_1)] - (\partial_t U^n(t_2) - N[u(t_2)])$ in place of $\partial_t U^n - N[u]$ evaluated at a single time. And now $\beta$ solves

\[
\begin{cases}
\Delta \beta = \overline{q} \overline{q}^{-2} & \text{in } \Omega, \\
\nabla \beta \cdot n = 0 & \text{on } \Gamma.
\end{cases}
\]

In place of (10.4), we find

\[
\|q(t_1) - q(t_2)\|_{L^r(\Omega)} = -\langle u(t_1) \otimes u(t_1) - u(t_2) \otimes u(t_2), \nabla \nabla \beta \rangle - \int_{\Gamma} U^n((u(t_1) - u(t_2)) \cdot \nabla \beta),
\]

where we note that the boundary integral involving $\partial_t U^n \beta$ appearing in (10.4) cancels.

For the first term on the right-hand side of (10.6), we use that

\[
\|u(t_1) \otimes u(t_1) - u(t_2) \otimes u(t_2)\|_{L^r(\Omega)} \leq \|u(t_1)\|_{L^\infty(\Omega)} + \|u(t_2)\|_{L^\infty(\Omega)} \|u(t_1) - u(t_2)\|_{L^r(\Omega)}.
\]

But, applying Lemma A.8 with $N = 0$,

\[
\|u(t_1) - u(t_2)\|_{L^r(\Omega)} \leq C \|u(t_1) - u(t_2)\|_{L^\infty(\Omega)} \leq C \|u\|_{C^\alpha(\Omega)} |t_1 - t_2|^{\alpha'},
\]

so

\[
-(u(t_1) \otimes u(t_1) - u(t_2) \otimes u(t_2), \nabla \nabla \beta) \leq C \|u\|_{L^\infty(\Omega)} \|u\|_{X_{\alpha', \alpha}} |t_1 - t_2|^{\alpha'} \|\beta\|_{W^2, r(\Omega)}.
\]

For the boundary integral in (10.6), we obtain as in (10.5),

\[
-\int_{\Gamma} U^n((u(t_1) - u(t_2)) \cdot \nabla \beta) \leq C \|u\|_{L^\infty(\Omega)} \|u(t_1) - u(t_2)\|_{L^\infty(\Omega)} \|\beta\|_{W^2, r(\Omega)}.
\]

Combining these bounds, we see that

\[
\|q(t_1) - q(t_2)\|_{L^r(\Omega)} \leq C \|u\|_{L^\infty(\Omega)} + \|u\|_{C^\alpha(\Omega)} \|u\|_{X_{\alpha', \alpha}} |t_1 - t_2|^{\alpha'},
\]

which is (10.3).

\[
(10.8)
\]

**Proposition 10.3.** Assume that the data has regularity $N$ and let $\Omega'$ be as in Lemma 10.1. Let $u \in X_{\alpha', \alpha}^{N+1}$ and let $q$ solving (3.6) be as in Lemma 10.1, we have

\[
\|q(t_1) - q(t_2)\|_{W^{N+2, r}(\Omega)} \leq C \|\Delta q\|_{L^\infty(\Omega)} + \|\nabla q \cdot n\|_{W^{N+1-\frac{1}{r}, r}(\Gamma)} + \|q\|_{L^r(\Omega)}.
\]

Now,

\[
\Delta \overline{q} = \nabla u(t_2) \cdot (\nabla u(t_2))^T - \nabla u(t_1) \cdot (\nabla u(t_1))^T = \nabla (u(t_2) - u(t_1)) \cdot (\nabla (u(t_2) - u(t_1)))^T.
\]

Thus, for $N = 0$,

\[
\|\Delta \overline{q}\|_{L^r(\Omega)} \leq C \|\nabla (u(t_1) - u(t_2))\|_{L^r(\Omega)} \left[ \|\nabla u(t_1)\|_{L^\infty(\Omega)} + \|\nabla u(t_2)\|_{L^\infty(\Omega)} \right].
\]

For $N \geq 1$, since $Nr > 3N \geq 3$, $W^{N, r}$ is an algebra, so

\[
\|\Delta \overline{q}\|_{W^{N, r}(\Omega)} \leq C \|\nabla (u(t_1) - u(t_2))\|_{W^{N, r}(\Omega)} \left[ \|\nabla u(t_1)\|_{W^{N, r}(\Omega)} + \|\nabla u(t_2)\|_{W^{N, r}(\Omega)} \right].
\]
In either case, we have
\[ \| \Delta \eta \|_{W^{N,r}(\Omega)} \leq C \| \nabla u \|_{L^\infty(0,T; W^{N,r}(\Omega))} \| \nabla (u(t_1) - u(t_2)) \|_{W^{N,r}(\Omega)}. \]

But,
\[ u(t_1) - u(t_2) = K_{U^n}[\omega(t_1)] - K_{U^n}[\omega(t_2)] = K[\omega(t_1) - \omega(t_2)] + w, \]
where
\[ w = \mathcal{V}(t_1) = \mathcal{V}(t_2) + u_c(t_1) - u_c(t_2). \]

Hence, applying Lemma 7.3,
\[ \| \nabla u(t_1) - \nabla u(t_2) \|_{W^{N,r}(\Omega)} \leq C \| \omega(t_1) - \omega(t_2) \|_{W^{N,r}(\Omega)} + C \| w \|_{W^{N,r}(\Omega)}. \]

Applying Lemma A.8,
\[ \| \omega(t_1) - \omega(t_2) \|_{W^{N,r}(\Omega)} \leq \| \omega(t_1) - \omega(t_2) \|_{C(\Omega)} \leq \| \omega \|_{C^{N,\alpha}(Q)} |t_1 - t_2|^\alpha. \]

Using Lemma A.8 again,
\[ \| w \|_{W^{N,r}(\Omega)} \leq C \| w \|_{C(\Omega)} \leq \| w \|_{C^{N,\alpha}(Q)} |t_1 - t_2|^\alpha \]
\[ \leq \| \mathcal{V} \|_{C^{N,\alpha}(Q)} |t_1 - t_2|^\alpha + \| u \|_{L^\infty(0,T; H)} |t_1 - t_2|^\alpha \leq c_X |t_1 - t_2|^\alpha, \]
where we also used Lemma 7.1. Hence,
\[ \| \nabla u(t_1) - \nabla u(t_2) \|_{W^{N,r}(\Omega)} \leq c_X |t_1 - t_2|^\alpha. \]

On \( \Gamma_+ \),
\[ \nabla \eta \cdot n = \partial_t U^n(t_1) - \partial_t U^n(t_2) + N[u(t_2)] - N[u(t_1)], \]
and we can see from the expression for \( N[u] \) in (9.3)—the key point being that on \( \Gamma_+ \), \( N[u] \) has no derivatives on \( u^T \)—that applying Lemma A.8 again,
\[ \| \nabla \eta \cdot n \|_{W^{N+\frac{1}{r},r}(\Gamma_+)} \leq \left[ \| \mathcal{U} \|_{C^{N+2,\alpha}(Q)}^2 + \| u \|_{C^{N+1,\alpha}(Q)} \| \mathcal{U} \|_{C^{N+1,\alpha}(Q)} \right] |t_1 - t_2|^\alpha \]
\[ \leq c_X |t_1 - t_2|^\alpha, \]
where in the last inequality we used a bound like that in (10.11).

Along with Proposition 10.2, these bounds give (10.8)₁.

Since we set \( r = 3/(1 - \alpha) \), Sobolev embedding gives \( W^{1,r}(\Omega') \subseteq C^{\alpha}(\Omega') \). Applying (10.8)₁ gives (10.8)₂.

**Remark 10.4.** It is only in the bound on \( \| \nabla \eta \cdot n \|_{W^{N+\frac{1}{r},r}(\Gamma_+)} \) in the proof of Proposition 10.3 that we use the higher regularity of \( \mathcal{U} \) over that of \( u \).

To account for time derivatives \( \partial_t^k q \), \( k \leq N + 1 \), we note that (3.6) becomes
\[
\begin{cases}
\Delta \partial_t^k q = -\partial_t^k (\nabla u \cdot (\nabla u)^T) & \text{in } \Omega, \\
\nabla \partial_t^k q \cdot n = -\partial_t^{k+1} U^n - \partial_t^k N[u] & \text{on } \Gamma,
\end{cases}
\]
and the same analysis in Propositions 10.2 and 10.3 applies to \( \partial_t^k q \). Then, from the key bound in (10.8), letting \( Q' = [0, T] \times \Omega' \), we have
\[ \| q(t_1) - q(t_2) \|_{C^{N+1,\alpha}(Q')} \leq c_X |t_1 - t_2|^\alpha. \]
Moreover, applying the interpolation inequality in Lemma A.4 and Proposition 10.2, we have,
\[ \|q(t_2) - q(t_1)\|_{C^{N+1}([0,T] \times \Gamma_+)} \leq C \|q(t_1) - q(t_2)\|_{C^{-1,\alpha}(Q)}^{\alpha} \|q(t_2) - q(t_1)\|_{L^2(\Omega)}^{1-a} \]
\[ \leq C \|c_X|t_1 - t_2|\|^{a'} \left[ c_X M|t_1 - t_2|^{a''} \right]^{1\!-\!a} \leq c_X|t_2 - t_1|^\alpha, \]
where \( \alpha < \alpha'' \approx \alpha + \alpha'(1-a) < \alpha' \) (using the value of \( \alpha \) for \( N+1 \) in Lemma A.4). In the second inequality, we applied Proposition 10.2 and (10.12).

Then from (10.12) and (10.13) and using that \( \|\nabla q(t_1) - \nabla q(t_2)\|_{C^0(\Gamma_+)} \leq \|\nabla q(t_1) - \nabla q(t_2)\|_{C^0(\Gamma_+)} \), we can apply Lemma A.7 with \( F_1(t) = c_X t^{\alpha} \), \( F_2(t) = c_X t^{\alpha''} \)
to obtain
\[ \|\nabla \Gamma q(t)\|_{C^{N,\alpha}(\{0,T\} \times \Gamma_+)} \leq \|\nabla q(0)\|_{C^{N,\alpha}(\Gamma_+)} + c_X T^{\alpha''} + c_X T^{\alpha''-\alpha} \leq c_0 + c_X T^{\alpha''}. \]

We used here that \( \nabla q(0) \) depends only upon the initial data along with Remark 6.2.

**PART 3: ESTIMATES ON THE OPERATOR A**

11. An invariant set

Recalling Remark 4.2, to define precisely the operator \( A \) described in Section 4, first define the operator \( \Lambda: \hat{C}_N^{N+1,\alpha}(Q) \rightarrow C^{N,\alpha}(Q) \) by \( \Lambda u = \omega \), where \( \omega \) is the solution to (2.1) given by Theorem 2.2 with \( H \) as in (3.7). Proposition 3.5 shows that (2.2) is satisfied, so \( \omega \) remains in the range of the curl. Hence, we can then let \( A u = v \), also from Theorem 2.2.

We now make a series of estimates leading in Proposition 5.1 to the existence of an invariant set in \( X_{\alpha',\alpha}^N \) for the operator \( A \).

**Proposition 11.1.** Assume that for \( N > 0 \) the data has regularity \( N \), \( \text{cond}_N \) holds, and that \( u \in X_{\alpha',\alpha}^N \). Then
\[ \|H\|_{L^\infty(\{0,T\} \times \Gamma_+)} \leq \|\omega_0\|_{L^\infty(\Gamma_+)} + M T^{\alpha'} \leq c_0 + M T^{\alpha'}, \]
\[ \|H\|_{C^{N,\alpha}(\{0,T\} \times \Gamma_+)} \leq c_0 + c_X T^{\alpha'}, \]
where \( a = \min\{\alpha, \alpha'' - \alpha\} > 0 \) (\( \alpha'' \) is as in (10.13)).

**Proof.** We have,
\[ \|H\|_{L^\infty(\{0,T\} \times \Gamma_+)} \leq \|H((t,x) - H(0,x))\|_{L^\infty(\{0,T\} \times \Gamma_+)} + \|H(0,x)\|_{L^\infty(\Gamma_+)} \]
\[ \leq \sup_{[0,T] \times \Gamma_+} |H(t,x) - H(0,x)| + \|\omega_0\|_{L^\infty(\Gamma_+)} \]
\[ \leq \|H\|_{C^0(\{0,T\} \times \Gamma_+)} T^{\alpha'} + \|\omega_0\|_{L^\infty(\Gamma_+)} \leq \|\omega\|_{C^0(Q)} T^{\alpha'} + \|\omega_0\|_{L^\infty(\Gamma_+)}. \]

We used \( \text{cond}_0 \) here in a subtle way, for it gave that \( H = \omega|_{\Gamma_+} \) down to time zero.

From (3.7), we can write,
\[ H^T = \delta_1 + \delta_2 - \nabla q, \quad H^u = \text{curl}_T \mathbf{U}^T, \]
where
\[ \delta_1 := \frac{1}{U n} \left[ -\partial_i \mathbf{U}^T - \nabla q \left( \frac{1}{2} |\mathbf{U}|^2 \right) + f \right]^\perp, \quad \delta_2 := \frac{1}{U n} \text{curl}_T \mathbf{U}^T \mathbf{U}^T. \]
Since \( \mathbf{U} \in C_{\sigma}^{N+2,\alpha}(Q) \), we see that \( \|\delta_1\|_{C^{N,\alpha}(\{0,T\}\times\Gamma_+)} \leq c_0 \) and, applying Corollary A.9,
\[
\|\delta_2\|_{C^{N,\alpha}(\{0,T\}\times\Gamma_+)} \leq C\|\mathbf{u}_0\|_{C^N(\{0,T\}\times\Gamma_+)} + C\|\mathbf{u}_T\|_{C^{N,\alpha}(\{0,T\}\times\Gamma_+)} T^\alpha
\]
\[
\leq c_0 + C\|\mathbf{u}\|_{C^{N,\alpha}(Q)} T^\alpha \leq c_0 + C\|\mathbf{u}\|_{X_{N,\alpha}^\prime} T^\alpha \leq c_0 + c_X T^\alpha.
\]
With (10.14), then, we see that
\[
\|\mathbf{H}\|_{C^{N,\alpha}(\{0,T\}\times\Gamma_+)} \leq c_0 + c_X T^\alpha.
\]
\[\square\]

The estimate in Proposition 11.2 is the analog (in terms of vorticity) of the key estimate in Equation (22) of [20].

**Proposition 11.2.** Assume that the data has regularity \( N \geq 0 \) and that \( \mathbf{u} \in X_{N,\alpha}^\prime \). Then
\[
\|\mathbf{u}\|_{L^\infty(Q)} \leq \|\omega_0\|_{L^\infty(\Omega)} + M T^\alpha e^{MT} \leq \|\omega_0\|_{L^\infty(\Omega)} e^{MT},
\]
\[
\|\mathbf{u}\|_{C^{N,\alpha}(Q)} \leq (1 + c_0) F_e(M,T) + c_X T^\alpha,
\]
for some \( a > 0 \) and \( F_e \) is continuous and increasing in its arguments, with \( F_e(M,0) = c_0 \).
(Note that \( \mathbf{u} = \text{curl} \mathbf{u} \).)

**Proof.** First assume no forcing. Let \( \omega_0 = \omega(0) \) and recall the definition of \( \gamma_0 \) in (8.5). From (8.6), we can write, \( \omega := \Lambda \mathbf{u} = \omega_\pm \) on \( U_\pm \), where
\[
\omega_-(t,x) = \nabla \eta(0,t;\gamma_0) \omega_0(\gamma_0) \text{ on } U_-,
\]
\[
\omega_+(t,x) = \nabla \eta(\tau(t,x),t;\gamma(t,x)) \mathbf{H}(\tau(t,x),\gamma(t,x)) \text{ on } U_+.
\]
\[\text{(11.1)}\]

It follows, using Lemma 8.2 and Proposition 11.1, that
\[
\|\omega_-(t,x)\|_{L^\infty(U_-)} \leq \|\nabla \eta\|_{L^\infty(\Omega)} \|\omega_0\|_{L^\infty(\Omega)} \leq \|\omega_0\|_{L^\infty(\Omega)} e^{MT},
\]
\[
\|\omega_+(t,x)\|_{L^\infty(U_+)} \leq \|\nabla \eta\|_{L^\infty(\Omega)} \|\mathbf{H}\|_{L^\infty(\{0,T\}\times\Gamma_+)} \leq \|\omega_0\|_{L^\infty(\Gamma_+)} + M T^\alpha e^{MT},
\]
which yields our bound on \( \|\Lambda \mathbf{u}\|_{L^\infty(Q)} \).

Let us now first treat the case \( N = 0 \), to get a better understanding of the estimates involved. Using Lemma 8.2 along with Lemmas A.1 and A.2, we see that
\[
\|\omega_-(t,x)\|_{C^{\alpha}(U_-)} \leq \|\nabla \eta(0,t;\gamma_0)\|_{C^{\alpha}(U_-)} \|\omega_0(\gamma_0)\|_{C^{\alpha}(U_-)}
\]
\[
\leq \|\nabla \eta(0,t;\gamma_0)(t_1,t_2;\mathbf{x})\|_{C^{\alpha}(\{0,T\}\times\Omega)} \|\omega_0(\gamma_0)\|_{C^{\alpha}(\Omega)} \leq \|\omega_0\|_{C^{\alpha}(\Omega)} [1 + \|D \mu\|_{L^\infty(Q)}] \alpha
\]
\[
\leq [c_0 + c_X T^\alpha] e^{2MT}.
\]
Similarly,
\[
\|\omega_+(t,x)\|_{C^{\alpha}(U_+)} \leq \|\nabla \eta(\tau(t,x),t;\gamma(t,x))\|_{C^{\alpha}(U_+)} \|\mathbf{H}(\tau(t,x),\gamma(t,x))\|_{C^{\alpha}(U_+)}
\]
\[
\leq \|\nabla \eta(\tau(t,x),t;\gamma(t,x))\|_{C^{\alpha}(\{0,T\}\times\Gamma_+)} [1 + \|D \mu\|_{L^\infty(Q)}] \alpha
\]
\[
\leq [c_0 + c_X T^\alpha] e^{2MT}.
\]

Again using Lemma 8.2, we see that
\[
\|\omega_+(t,x)\|_{C^{\alpha}(U_+)} \leq [e^{MT} + e^{(1+2\alpha)MT} M T^{1-\alpha}] [c_0 + c_X T^\alpha] [1 + \|D \mu\|_{L^\infty(Q)}] \alpha
\]
Recalling, from the comment following Definition 6.1, that
\[ ||u||_{L^\infty(Q)} \leq c_0 + [c_0 + MT^\alpha]e^{MT} + CMT^b, \]
so (8.2) gives
\[ 1 + ||D\mu||_{L^\infty(Q)} \leq c_0[1 + ||u||_{L^\infty(Q)}]^2e^{2MT} \leq c_0[|c_0 + MT^\alpha|^2e^{2MT}]e^{MT} + CMT^b. \]
Hence,
\[ ||\omega_+(t,x)||_{C^\alpha(U_+)} \leq c_0[e^{MT} + e^{(1+2\alpha)MT}MT^{1-\alpha}][c_0 + c_XT^\alpha][|c_0 + MT^\alpha|^4e^{4\alpha MT} + M^4T^{4b}]. \]
But we know from Theorem 2.2 that \( \omega \in C^\alpha(Q) \), because we assumed \( \text{cond}_0 \); hence, taking the maximum of the bounds on \( U_+ \) leads to a bound of the form,
\[ ||Au||_{C^\alpha(Q)} \leq (1 + c_0)F_c(M,T) + c_XT^{a'}, \]
where \( a' > 0 \), and where \( F_c(M,0) = c_0 \). Including forcing only adds a \( c_XT \) term to the bound, as we can see from (8.6), so an estimate of the same form holds with forcing.

Now consider \( N \geq 1 \). The expressions for \( \omega_\pm \) in (11.1) each consist of two factors. We first apply Leibniz's product rule to these expressions then apply the chain rule to each term. For \( \omega_+ \), if \( \beta \) is a time-space multi-index with \( |\beta| = N \), then \( D^\beta\omega_+ \) consists of a finite sum of terms of the form,
\[ D^{\beta_1}\nabla\eta(t,x), \gamma(t,x))D^{\beta_2}\mathbf{H}(\tau(t,x)) \prod_{\ell=1}^n D^{\beta_{\ell}}\mu(t,x) \text{ on } U_+, \]
where \( \beta_1 + \beta_2 = \beta \) and \( \sum_{\ell=1}^n|\beta_\ell|^2 = |\beta| \). The factors can be controlled by Proposition 11.1, Lemma 8.2, and (7.2). Following the similar process for \( D^\beta\omega_- \) leads to an estimate for \( ||Au||_{C^{N,\alpha}(Q)} \) of the same form as for \( ||Au||_{C^\alpha(Q)} \).

Having established our many estimates, we can now give the proof of Proposition 5.1.

Proof of Proposition 5.1. Because we are assuming that \( u(0) = u_0 \), step (0) in the definition of \( A \) can be ignored. We have, from Proposition 11.2,
\[ ||Au||_{C^{N,\alpha}(Q)} \leq (1 + c_0)F_c(M,T) + c_XT^a. \]
Recalling, from the comment following Definition 6.1, that \( c_0 \) may increase with \( T \), let \( c_0(0) \geq 0 \) be its value for \( T = 0 \). Start by choosing any
\[ M > M_0 := \max\{(3(1 + c_0(0))\frac{1}{2}, 3\|P_{H_u}u_0\|_{C^{N+1,\alpha}(\Omega)}, 1\}, \]
which gives \( (1 + c_0(0))F_c(M,0) < M/3 \). Next, by continuity there exists \( T > 0 \) such that
\[ (1 + c_0)F_c(M,T) \leq \frac{M}{3}. \]
We can choose \( T > 0 \) small enough that
\[ c_XT^a \leq \frac{M}{3}. \]
It follows that
\[ ||\text{curl } Au||_{C^{N,\alpha}(Q)} \leq \frac{2M}{3}. \]
Then, because \( M > 3\|P_{H_u}u_0\|_{C^{N+1,\alpha}(\Omega)} \), we see from (2.6) that \( ||Au||_{X_{\alpha'},\alpha} \leq M \), after again decreasing \( T \) if necessary. \( \square \)
To prove Proposition 5.2, we first make some definitions and establish a few lemmas.

Throughout this section, we let $M$, $T$, and $K$ be given as in Proposition 5.1. We assume that $u_1, u_2$ are two vector fields in $K$ and, for $j = 1, 2$, we let $ω_j = \text{curl} u_j$, with $η_j, τ_j, γ_j$, $U^j_±$, and $S_j$ defined for the velocity field $u_j$. We let $V_± = U^j_± \cap U^j_±$, $W = Q \setminus (V_+ \cup V_-)$.

By virtue of Lemma 8.2, we have, for $j = 1, 2$,
\[ \|η_j(0, \cdot, \cdot)\|_{C^{N+1, α}(Q)} \leq C(T, M). \] 

The dependence of constants on $T$ and $M$ will have no impact on the proof of Proposition 5.2, since they are fixed. We include such dependence, however, when we judge that it makes the nature of the bound being derived clearer.

We define $μ_j: U_+ \rightarrow [0, T] × Γ_+$ by $μ_j(t, x) = (τ_j(t, x), γ_j(t, x))$. We let
\[ w := u_1 - u_2, \quad μ := μ_1 - μ_2. \]

We fix $β \in (0, α]$ arbitrarily and let
\[ θ_β := \|w\|_{X_{β, β}} = \|w\|_{C^β(Q)} + \|\text{curl } w\|_{C^β(Q)}. \] 

**Lemma 12.1.** We have,
\[ \|μ\|_{L^∞(V_+)} \leq C(T, M)Tθ_β. \]

*Proof.* We know from Lemma 3.5 of [12] that $μ_j$ is transported by the flow map for $u_j$; that is,
\[ \partial_t μ_1 + u_1 \cdot ∇μ_1 = 0, \]
\[ \partial_t μ_2 + u_2 \cdot ∇μ_2 = 0. \]

Hence,
\[ \partial_t μ + u_1 \cdot ∇μ = -w \cdot ∇μ_2, \]
or,
\[ \frac{d}{dt}μ(t, η_1(0, t; x)) = -(w \cdot ∇μ_2)(t, η_1(0, t; x)) \leq \|w\|_{L^∞(Q)} \|∇μ_2\|_{L^∞(Q)} \leq C(T, M)θ_β, \]

where we used Lemma 8.2. Since $μ(t, η_1(0, t; x))|_{t=0} = 0$, our desired bound follows. \hfill \Box

**Lemma 12.2.** We have
\[ \|η_1 - η_2\|_{L^∞([0, T]×Ω)} \leq C(T, M)Tθ_β, \]
\[ \|∇η_1 - ∇η_2\|_{L^∞([0, T]×Ω)} \leq C(T, M)[θ_β + θ_β^2]. \]

*Proof.* We have,
\[ η_1(t_1, t_2; x) - η_2(t_1, t_2; x) = \int_{t_1}^{t_2} [u_1(s, η_1(t_1, s; x)) - u_2(s, η_2(t_1, s; x))] \, ds. \]

Fixing $t_1$, using (12.1), Lemma A.2, Lemma A.3, and applying Minkowski’s integral inequality gives
\[ |η_1(t_1, t; x) - η_2(t_1, t; x)| \]
\[ \leq \int_{t_1}^{t} \|u_1(s, η_2(t_1, s; x)) - u_2(s, η_2(t_1, s; x))\|_{L^∞} \, ds \]
\[
+ \int_{t_1}^{t} \| u_1(s, \eta_1(t_1, s; x)) - u_1(s, \eta_2(t_1, s; x)) \|_{L^\infty} \, ds
\]
\[
\leq \int_{t_1}^{t} \| u_1(s) - u_2(s) \|_{L^\infty} \, ds + \int_{t_1}^{t} \| u_1(s) \|_{C^1} \| \eta_1(t_1, s; x) - \eta_2(t_1, s; x) \|_{L^\infty} \, ds
\]
\[
\leq T \theta_\beta + C(T, M) \int_{t_1}^{t} \| \eta_1(t_1, s; x) - \eta_2(t_1, s; x) \|_{L^\infty} \, ds.
\]
Taking the supremum over \( x \) and applying Grönwall’s lemma gives
\[
\| \eta_1(t_1, t; x) - \eta_2(t_1, t; x) \|_{C([0, T]; L^\infty(\Omega))} \leq T e^{C(T, M)T \theta_\beta}.
\]
Since this holds uniformly for all \( t_1, t \in [0, T] \), we obtain the first bound.

Similarly, starting from
\[
\nabla \eta_1(t_1, t; x) - \nabla \eta_2(t_1, t; x) = \int_{t_1}^{t} [\nabla_x (u_1(s, \eta_1(t_1, s; x))) - \nabla_x (u_2(s, \eta_2(t_1, s; x)))] \, ds
\]
\[
= \int_{t_1}^{t} [\nabla u_1(s, \eta_1(t_1, s; x)) \nabla \eta_1(t_1, s; x) - \nabla u_2(s, \eta_2(t_1, s; x)) \nabla \eta_2(t_1, s; x)] \|_{L^\infty} \, ds,
\]
we find
\[
|\nabla \eta_1(t_1, t; x) - \nabla \eta_2(t_1, t; x)|
\]
\[
\leq \int_{t_1}^{t} \| \nabla u_1(s, \eta_1(t_1, s; x)) \|_{C^0} \| \eta_1(t_1, s; x) - \eta_2(t_1, s; x) \|_{L^\infty} \, ds
\]
\[
+ \int_{t_1}^{t} \| \nabla u_1(s, \eta_2(t_1, s; x)) - \nabla u_2(s, \eta_1(t_1, s; x)) \|_{L^\infty} \, ds
\]
\[
+ \int_{t_1}^{t} \| \nabla u_2(s, \eta_2(t_1, s; x)) \|_{L^\infty} \, ds
\]
\[
\leq C(T, M) [T e^{C(T, M)T \theta_\beta}]^\alpha T + C(M, T) T \theta_\beta
\]
\[
+ C(M, T) \int_{t_1}^{t} \| \nabla \eta_1(t_1, s; x) - \nabla \eta_2(t_1, s; x) \|_{L^\infty} \, ds.
\]
In the last inequality, we used Lemma 7.3 to conclude that \( \| \nabla u_1(s) - \nabla u_2(s) \|_{L^\infty(\Omega)} \leq \| \nabla w(s) \|_{C^{1, \beta}(\Omega)} \leq C \| \text{curl} w(s) \|_{C^{\beta}(\Omega)} + C \| w(s) \|_H \leq C \theta_\beta \). Taking the supremum over \( x \) and applying Grönwall’s lemma as before gives the second bound.

**Lemma 12.3.** Letting \( |W| \) be the Lebesgue measure of \( W := Q \setminus (V_+ \cup V_-) \), we have
\[
|W| \leq C(T, M) T^2 \theta_\beta.
\]

**Proof.** The set \( W(t) := \{ x \in \Omega : (t, x) \in W \} \) consists of all points lying between \( S_1(t) \) and \( S_2(t) \). Any \( x_1 \in S_1(t) \) is of the form \( x_1 = \eta_1(0, t; y) \) for some \( y \in \Gamma_+ \), and by Lemma 12.2,
the point \( x_2 = \eta_2(0, t; y) \) is within a distance \( \delta = C(T, M)T\theta_\beta \) of \( x_1 \). That is, any point in \( S_1(t) \) is within a distance \( \delta \) of \( S_2(t) \) and the relation is symmetric. So

\[
W(t) \subseteq W_\delta(t) := \{x \in \Omega: \text{dist}(x, S^1(t)) \leq \delta\}.
\]

As we observed in Section 8, \( S^1(t) \) is at least \( C^{1,\alpha} \) regular as a surface in \( \Omega \), and so has finite perimeter, so we can see that \( |W_\delta(t)| \leq C\delta \). Moreover, this constant can depend upon \( T \) and \( M \), but is bounded over \( T \), for as also observed in Section 8, \( S^1 \) is at least \( C^{1,\alpha} \) regular as a hypersurface in \( Q \). Thus, \( |W| \leq T|W_\delta(t)| \leq C(T, M)T^2\theta_\beta \). \( \square \)

**Proof of Proposition 5.2.** Let \( u_1, u_2 \in K \). We will obtain a bound in the following three steps:

(A) Bound the difference in vorticities, \( \Lambda u_1 - \Lambda u_2 \), assuming zero forcing.

(B) Account for forcing in the bound on \( \Lambda u_1 - \Lambda u_2 \).

(C) Account for the harmonic component of \( u_1 \) and \( u_2 \) to bound \( u_1 - u_2 \).

**A)** Vorticity: The key to this proof is Lemma A.5, written in the form,

\[
\|f\|_{C^{N,\beta}(Q)} \leq \|f\|_{L^\infty(Q)} + F(\|f\|_{C^{N,\alpha}(Q)})\|f\|_{L^2(Q)}^{1-a},
\]

where \( F(x) = x^{a_1} + x^{a_\alpha} + x^{a'} \), where \( a_n \) is given in Lemma A.4 and \( a' \) in Lemma A.5. The exponent \( a \) would depend upon whether \( \|f\|_{L^2(Q)} \) is greater or less than 1. Applied to \( f := \Lambda u_1 - \Lambda u_2 \), we have

\[
\|\Lambda u_1 - \Lambda u_2\|_{C^{N,\beta}(Q)} \leq \|\Lambda u_1 - \Lambda u_2\|_{L^\infty(Q)} + C(M)\|\Lambda u_1 - \Lambda u_2\|_{L^2(Q)}^{1-a},
\]

since \( F(\|\Lambda u_1 - \Lambda u_2\|_{C^{N,\alpha}(Q)}) \leq M^{a_1} + M^{a_\alpha} + M^{a'} \leq C(M) \). We conclude that to prove the continuity of \( \Lambda \) in the \( C^{N,\beta}(Q) \) norm it suffices to obtain a bound on \( \Lambda u_1 - \Lambda u_2 \) in \( L^\infty(Q) \).

Letting \( (t, x) \in Q \), we must estimate \( |\Lambda u_1(t, x) - \Lambda u_2(t, x)| \). This involves three cases: (1) \( (t, x) \in V_- \), (2) \( (t, x) \in V_+ \), (3) \( (t, x) \in W \), which we consider separately. We argue first without forcing.

(1) Define, for \( (t, x) \in V_- \), \( j = 1, 2 \),

\[
\gamma_0^j = \gamma_0^j(t, x) := \eta_j(t, 0; x).
\]

From (8.6), we can write,

\[
\Lambda u_1(t, x) - \Lambda u_2(t, x) = \nabla \eta_1(0, t; \gamma_0) \omega_0(\gamma_0) - \nabla \eta_2(0, t; \gamma_0) \omega_0(\gamma_0) = I_1 + I_2,
\]

where

\[
I_1 := \omega_0(\gamma_0^1) \cdot (\nabla \eta_1(0, t; \gamma_0) - \nabla \eta_2(0, t; \gamma_0^2)),
\]

\[
I_2 := (\omega_0(\gamma_0^1) - \omega_0(\gamma_0^2)) \cdot \nabla \eta_2(0, t; \gamma_0^2).
\]

We also make the decompistion, \( I_1 = \omega_0(\gamma_0^1) \cdot (I_1^1 + I_1^2) \), where

\[
I_1^1 := \nabla \eta_1(0, t; \gamma_0^1) - \nabla \eta_1(0, t; \gamma_0^2),
\]

\[
I_1^2 := \nabla \eta_2(0, t; \gamma_0^1) - \nabla \eta_2(0, t; \gamma_0^2).
\]

Then,

\[
\|I_1\|_{L^\infty(V_-)} \leq \|\omega_0\|_{L^\infty(\Omega)} \left(\|I_1^1\|_{L^\infty(V_-)} + \|I_1^2\|_{L^\infty(V_-)}\right).
\]

with

\[
\|I_1^1\|_{L^\infty(V_-)} \leq \|\nabla \eta_1(0, t; x)\|_{L^\infty(\Omega)} \|\eta_1(t, 0; \cdot) - \eta_2(t, 0; \cdot)\|_{L^\infty(\Omega)} \leq C(T, M)T[\theta_\beta]^\alpha \leq C(T, M)T^{1+\alpha} \theta_\beta^2,
\]
where in the second-to-last equality we added the bounds on $H$ and also used cond
\[ \| \nabla \eta_1 (t, x) \|_{L^\infty(V_-)} \leq C(T, M) T [\theta_\beta + \theta_\beta^0], \]
where we applied Lemma 12.2. And, applying Lemmas 12.2 and A.3,
\[ \| I_2 \|_{L^\infty(V_-)} \leq \| \omega_0 \|_{C^0(\Omega)} \| \eta_1 (t, 0, \cdot) - \eta_2 (t, 0, \cdot) \|_{L^\infty(V_-)}^2 \| \nabla \eta_2 (0, t, \cdot) \|_{L^\infty(V_-)} \]
\[ \leq C(T, M^0) M [C(T, M) T \theta_\beta]^8. \]

Dropping the dependence upon $M$ or the initial data, which play no role here, we conclude
\[ \| \Lambda u_1 (t, x) - \Lambda u_2 (t, x) \|_{L^\infty(V_-)} \leq C(T) [\theta_\beta + \theta_\beta^0]. \]

(2) For $(t, x) \in V_+$, we have
\[ \Lambda u_1 (t, x) - \Lambda u_2 (t, x) = H_1 (\mu_1 (t, x)) \cdot \nabla \eta_1 (\gamma_1 (t, x), t) - H_2 (\mu_2 (t, x)) \cdot \nabla \eta_2 (\gamma_2 (t, x), t) \]
\[ = J_1 + J_2 + J_3, \]
where $H_j (t, x)$ is from (3.7) and
\[ J_1 := H_1 (\mu_1 (t, x)) \cdot (\nabla \eta_1 (\tau_1 (t, x), t) - \nabla \eta_2 (\gamma_1 (t, x), t)), \]
\[ J_2 := H_2 (\mu_2 (t, x)) \cdot (\nabla \eta_1 (\tau_2 (t, x), t) - \nabla \eta_2 (\gamma_2 (t, x), t)), \]
\[ J_3 := (H_1 (\mu_1 (t, x)) - H_2 (\mu_2 (t, x))) \cdot \nabla \eta_2 (\gamma_2 (t, x), t). \]

Now, since $H_j (s, y) = \omega_j (s, y)$ for $(s, y) \in [0, T] \times \Gamma_+$, we have, using Lemma 12.2,
\[ \| J_1 \|_{L^\infty(V_+)} \leq \| \omega_1 \|_{L^\infty(Q)} \| \nabla \eta_1 (\cdot, t; \cdot) - \nabla \eta_2 (\cdot, t; \cdot) \|_{L^\infty(Q)} \leq C(T, M) [\theta_\beta + \theta_\beta^0], \]
where we also used condo. For $J_2$, we have, using Lemmas 12.1 and A.3,
\[ \| J_2 \|_{L^\infty(V_+)} \leq \| \omega_1 \|_{L^\infty(Q)} \| \nabla \eta_2 (\cdot, \cdot; \cdot) \|_{L^\infty(Q)} \leq C(T, M) [\theta_\beta + \theta_\beta^0]. \]

For $J_3$, we have
\[ J_3 \leq \| H_1 (\mu_1 (t, x)) - H_2 (\mu_2 (t, x)) \|_{L^\infty(U_+)} \| \nabla \eta_2 \|_{L^\infty(Q)}, \]
But, $\| \nabla \eta_2 \|_{L^\infty(Q)} \leq C(T, M)$ by Lemma 8.2, and, using Lemma A.3,
\[ \| H_1 (\mu_1 (t, x)) - H_2 (\mu_2 (t, x)) \|_{L^\infty(U_+)} \]
\[ \leq \| H_1 (\mu_1 (t, x)) - H_2 (\mu_1 (t, x)) \|_{L^\infty(U_+)} + \| H_2 (\mu_1 (t, x)) - H_2 (\mu_2 (t, x)) \|_{L^\infty(U_+)} \]
\[ \leq \| H_1 - H_2 \|_{L^\infty([0, T] \times \Gamma_+)} + \| \omega_1 - \omega_2 \|_{L^\infty([0, T] \times \Gamma_+)} \leq C(T, M) [\theta_\beta + \theta_\beta^0]. \]

where in the second-to-last equality we added the bounds on $H_1$ and $H_2$ coming from Proposition 11.1 and appealed to condo.

Combined, we see that
\[ \| \Lambda u_1 (t, x) - \Lambda u_2 (t, x) \|_{L^\infty(V_+)} \leq C(T, M) [\theta_\beta + \theta_\beta^0]. \]

(3) Now assume $(t, x) \in W$. Because $\Lambda u_1 - \Lambda u_2$ has a modulus of continuity given by its $C^0$ norm, which is bounded by $M$, it follows that
\[ \| \Lambda u_1 - \Lambda u_2 \|_{L^\infty(W)} \leq F_M (\| \Lambda u_1 - \Lambda u_2 \|_{L^2(W)}) \]
for a continuous function $F_M$ with $F_M (0) = 0$, as in Lemma A.10. From Lemma 12.3, then
\[ \| \Lambda u_1 - \Lambda u_2 \|_{L^2(W)} \leq \| \Lambda u_1 - \Lambda u_2 \|_{L^\infty(W)} |W|^2 \leq C M |W|^2 \leq C(T, M) \theta_\beta, \]
giving a bound on $\| \Lambda u_1 - \Lambda u_2 \|_{L^\infty(W)}$.
(B) Accounting for forcing: To treat forcing, let $G_{\pm}^2$ be defined for $\eta_j$ as $G_{\pm}$ is in (8.6). Then
\[
\|G_{\pm}^1 - G_{\pm}^2\|_{L^\infty(V_\pm)} 
\leq \int_0^T \|\nabla \eta_1(s, t; \eta_1(t, s; x))g(s, \eta_1(t, s; x)) - \nabla \eta_2(s, t; \eta_2(t, s; x))g(s, \eta_2(t, s; x))\|_{L^\infty(\Omega)} ds.
\]
But,
\[
\|\nabla \eta_1(s, t; \eta_1(t, s; x))g(s, \eta_1(t, s; x)) - \nabla \eta_2(s, t; \eta_2(t, s; x))g(s, \eta_2(t, s; x))\|_{L^\infty(\Omega)} 
\leq \|\nabla \eta_1(s, t; \eta_1(t, s; x))g(s, \eta_1(t, s; x)) - \nabla \eta_2(s, t; \eta_2(t, s; x))g(s, \eta_1(t, s; x))\|_{L^\infty(\Omega)} 
+ \|\nabla \eta_2(s, t; \eta_2(t, s; x))g(s, \eta_2(t, s; x)) - \nabla \eta_2(s, t; \eta_2(t, s; x))g(s, \eta_2(t, s; x))\|_{L^\infty(\Omega)} 
\leq \|\nabla \eta_1 - \nabla \eta_2\|_{L^\infty([0,T]^2 \times \Omega)}\|g\|_{L^\infty(Q)} + \|\nabla \eta_1 - \nabla \eta_2\|_{L^\infty([0,T]^2 \times \Omega)}\|g\|_{L^\infty(Q)} 
+ \|\nabla \eta_2\|_{L^\infty([0,T]^2 \times \Omega)}\|g\|_{L^\infty(Q)}
\]
where we used Lemmas A.2 and A.3.

Since $g \in L^\infty(Q)$, while $\nabla \eta_1$ and $\nabla \eta_2$ are bounded in $C^\alpha([0,T] \times \Omega)$, with the bounds in Lemma 12.2 we see—suppressing unimportant dependence of constants on $M$ and $T$—that
\[
\|G_{\pm}^1 - G_{\pm}^2\|_{L^\infty(V_\pm)} \leq CT[\theta_\beta + \theta_\beta^2].
\]
Hence, forcing does not change our bounds on $\|\Lambda u_1(t, x) - \Lambda u_2(t, x)\|_{L^\infty(V_\pm)}$ in (1), (2). And $G_{\pm}^1$, $G_{\pm}^2$ are bounded on $Q$, so the estimate on $\|\Lambda u_1 - \Lambda u_2\|_{L^2(V)}$ in (3) is also unchanged.

(C) Velocity: It remains to deal with the harmonic component of $v_1 - v_2$. Letting $\Omega_j$ be the matrix-valued function corresponding to $u_j$ in (2.6), we see that
\[
P_{H, \nu}(t) := P_{H, \nu}u_j(0) + \int_0^t P_{H, \nu}f(s) ds - \int_0^t P_{H, \nu}(\Omega_j(s)u_j(s)) ds.
\]
By Lemma 7.1, $\|P_{H, \nu}u\|_{L^\infty(Q)} \leq C\|u\|_H$ for any $u \in H$, so, noting that $v_1(t) - v_2(t) \in H$,
\[
\|P_{H, \nu}(v_1 - v_2)\|_{L^\infty(Q)} \leq \|u_1(0) - u_2(0)\|_H + \int_0^t \|P_{H, \nu}(\Omega_1u_1 - \Omega_2u_2)\|_H ds
\]
\[
\leq C\theta_\beta + \int_0^t \|\Omega_1u_1 - \Omega_2u_2\|_{L^\infty(Q)} ds.
\]
But,
\[
\int_0^t \|\Omega_1u_1 - \Omega_2u_2\|_{L^\infty(Q)} ds
\]
\[
\leq \int_0^t \|\Omega_1(s)(u_1 - u_2)(s)\|_{L^2(\Omega)} ds + \int_0^t \|\Omega_1 - \Omega_2\|_{L^2(\Omega)} ds
\]
\[
\leq \int_0^t \|\Omega_1(s)\|_{L^\infty(\Omega)}\|u_1 - u_2\|_{L^2(\Omega)} ds + \int_0^t \|\Omega_1 - \Omega_2\|_{L^\infty(\Omega)}\|u_2\|_{L^\infty(\Omega)} ds
\]
\[
\leq MT\theta_\beta,
\]
where we used that the nonzero components of $\Omega_j$ come from $\omega_j$.

Applying (12.3), we conclude that
\[
\|Au_1 - Au_2\|_{L^\infty(Q)} \leq C(M, T)[\|u_1 - u_2\|_{C^\beta(Q)} + \|u_1 - u_2\|_{C^\beta(Q)}],
\]
which shows that $A: K \rightarrow K$ is continuous in the $X_{\beta, \beta}$ norm.

13. Full inflow boundary condition satisfied

A key step in obtaining existence for inflow boundary conditions is showing in Proposition 5.3 that once we obtain a solution satisfying (1.9)$_{L,4}$ it also satisfies (1.9)$_{5}$. That this can be done by defining $H$ as in (3.7) and recovering the pressure using $N[u]$ as in (3.6) and (9.2) is one of the deep insights of [3].

**Proof of Proposition 5.3.** Our proof draws heavily from the proof of Lemma 4.2.1 pages 156-159 of [3]. Let 

$$w = u^\tau - U^\tau, \quad P := p - q.$$  

By Proposition 3.1, on $[0, T] \times \Gamma_+$, $\omega = W[u, p]$ of (3.5). From (1.2), (3.6), and (9.2), we see that on $\Gamma_+$, $\nabla P \cdot n = \text{div}_r(U^n w)$. Hence, $P$ satisfies

$$\begin{aligned}
\Delta P &= 0 \quad \text{in } \Omega, \\
\nabla P \cdot n &= 0 \quad \text{on } \Gamma_+ \cup \Gamma_0, \\
\nabla P \cdot n &= \text{div}_r(U^n w) \quad \text{on } \Gamma_+.
\end{aligned}$$

Multiplying by $P$ and integrating over $\Omega$ gives

$$\|\nabla P\|_{L^2(\Omega)}^2 = -(\Delta P, P) + \int_{\Gamma_+} (\nabla P \cdot n) P = \int_{\Gamma_+} \text{div}_r(U^n w) P = -\int_{\Gamma_+} U^n w \cdot \nabla P. \tag{13.1}$$

By (3.4) and the assumption that $H = \omega$ on $\Gamma_+$, we know that $U^n[H]^\perp = U^n[W]^\perp [u, p]^\perp$. Using also that $(v^\perp)^\perp = -v$, we have, from (3.5) and (3.7), that on $\Gamma_+$,

$$\partial_t U^\tau + \nabla F \left( q + \frac{1}{2} |U|^2 \right) - f^\tau + \text{curl}_r U^\tau [u^\tau]^\perp = H.$$  

Subtracting the left hand side from the right hand side, we have

$$0 = \nabla \cdot P + \frac{1}{2} \nabla F (|u|^2 - |U|^2) + \partial_t w + \text{curl}_r w [u^\tau]^\perp.$$  

But, $\omega^\perp = F^n$ on $\Gamma_+$, which gives $\text{curl}_r U^\tau = \text{curl}_r u^\tau$. Hence, $\text{curl}_r w = 0$, so

$$\nabla \cdot P = -\partial_t w - \frac{1}{2} \nabla (|u|^2 - |U|^2).$$

Returning to (13.1), we thus have

$$\|\nabla P\|_{L^2(\Omega)}^2 = \int_{\Gamma_+} U^n w \cdot \partial_t w + \frac{1}{2} \int_{\Gamma_+} U^n w \cdot \nabla (|u|^2 - |U|^2).$$

Now,

$$\int_{\Gamma_+} U^n w \cdot \partial_t w = \frac{1}{2} \int_{\Gamma_+} U^n |\partial_t w|^2 = \frac{1}{2} \int_{\Gamma_+} \partial_t [U^n |w|^2] - \frac{1}{2} \int_{\Gamma_+} \partial_t U^n |w|^2$$

$$= \frac{1}{2} \frac{d}{dt} \int_{\Gamma_+} U^n |w|^2 - \frac{1}{2} \int_{\Gamma_+} \partial_t U^n |w|^2,$$

so

$$\frac{d}{dt} \int_{\Gamma_+} U^n |w|^2 = \int_{\Gamma_+} \partial_t U^n |w|^2 - \int_{\Gamma_+} U^n w \cdot \nabla (|u|^2 - |U|^2) + 2 \|\nabla P\|_{L^2(\Omega)}^2. \tag{13.2}$$
Writing $|\mathbf{U}|^2 - |\mathbf{u}|^2 = |\mathbf{u}^T|^2 - |\mathbf{U}^T|^2 = \mathbf{w} \cdot \mathbf{v}$ on $\Gamma_+$, where $\mathbf{v} := \mathbf{U}^T + \mathbf{u}^T$, we have
\[
\int_{\Gamma_+} U^n \mathbf{w} \cdot \nabla \mathbf{r}(|\mathbf{u}|^2 - |\mathbf{U}|^2) = \int_{\Gamma_+} U^n \mathbf{w} \cdot \nabla \mathbf{r}(\mathbf{w} \cdot \mathbf{v})
= \int_{\Gamma_+} U^n (\mathbf{w} \cdot \nabla \mathbf{r}) \cdot \mathbf{w} + \int_{\Gamma_+} U^n (\mathbf{w} \cdot \nabla \mathbf{r}) \cdot \mathbf{v}
= \int_{\Gamma_+} U^n (\mathbf{w} \cdot \nabla \mathbf{r}) \cdot \mathbf{w} - \frac{1}{2} \int_{\Gamma_+} |\mathbf{w}|^2 \text{div}(U^n \mathbf{v}).
\]
For the last term above, we used that $U^n (\mathbf{w} \cdot \nabla \mathbf{r}) \cdot \mathbf{v} = (1/2) U^n \mathbf{v} \cdot \nabla \mathbf{r} |\mathbf{w}|^2$ and integrated by parts via Lemma B.1. Then because $\mathbf{v}$ and $U^n$ are sufficiently regular, we have
\[
\left| \int_{\Gamma_+} U^n \mathbf{w} \cdot \nabla \mathbf{r}(|\mathbf{u}|^2 - |\mathbf{U}|^2) \right| \leq C \int_{\Gamma_+} |\mathbf{w}|^2.
\]
Changing sign in (13.2) and integrating in time, we see that
\[
\int_{\Gamma_+} |U^n(t)| |\mathbf{v}(t)|^2 = - \int_{\Gamma_+} U^n(t) |\mathbf{w}(t)|^2 \\
\leq - \int_0^t \int_{\Gamma_+} \partial_t U^n |\mathbf{w}|^2 + \int_0^t \int_{\Gamma_+} U^n \mathbf{w} \cdot \nabla \mathbf{r}(|\mathbf{u}|^2 - |\mathbf{U}|^2) - 2 \int_0^t \|\nabla P\|_{L^2(\Omega)}^2 \\
\leq C \int_0^t \int_{\Gamma_+} |\mathbf{w}(s)|^2 \, ds - 2 \int_0^t \|\nabla P\|_{L^2(\Omega)}^2 \leq C \int_0^t \int_{\Gamma_+} |\mathbf{w}(s)|^2 \, ds,
\]
In the first equality we used that $U^n < 0$ on $\Gamma_+$, in the second equality we used that $\mathbf{w}(0) = 0$, and in the third equality we used that $\partial_t U^n$ is bounded.

Now using that $|U^n|$ is bounded away from zero, we have
\[
\int_{\Gamma_+} |\mathbf{w}(t)|^2 \leq C \int_0^t \int_{\Gamma_+} |\mathbf{w}(s)|^2 \, ds,
\]
and we conclude from Grönwall's lemma that $\mathbf{w} \equiv 0$. This means that $\mathbf{u}^T = \mathbf{U}^T$, so (1.9) holds. \hfill \Box

14. VORTICITY BOUNDARY CONDITIONS

**Proof of Theorem 1.3.** The proof of existence is the same as that for Theorem 1.2 except that $\mathbf{H}$ is given. Hence, there are no pressure estimates involved, so the estimates in Proposition 11.1 would simply be
\[
\|\mathbf{H}\|_{L^\infty([0,T] \times \Gamma_+)} \leq c_0, \quad \|\mathbf{H}\|_{C^{N,\alpha}([0,T] \times \Gamma_+)} \leq c_0,
\]
since $\mathbf{H}$, being given, can be treated as part of $c_0$. The condition in (1.17) immediately gives (2.2), so there is no need to appeal to Proposition 3.5. And since we only require $\mathbf{u} \cdot \mathbf{n} = U^n$ on $\Gamma_+$, there is no need to invoke Proposition 5.3. Otherwise, the remainder of the proof of existence proceeds unchanged.

For uniqueness when $N \geq 1$, let $\omega_j = \text{curl } \mathbf{u}_j$, $j = 1, 2$, and let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$. Then $\mathbf{w} \in H_0$, since $\mathbf{u}_1$, $\mathbf{u}_2$ have the same prescribed harmonic component, $\mathbf{u}_c$. Let $\mu = \text{curl } \mathbf{w} = \omega_1 - \omega_2$. Since $N \geq 1$, we have enough regularity to write $\partial_t \omega_j + \mathbf{u}_j \cdot \nabla \omega_j = \omega_j \cdot \nabla \mathbf{u}_j + \text{curl } \mathbf{f}$, and subtracting this relation for $j = 2$ from that for $j = 1$ gives
\[
\partial_t \mu + \mathbf{u}_1 \cdot \nabla \mu + \mathbf{w} \cdot \nabla \omega_2 = \omega_1 \cdot \nabla \mathbf{w} + \mu \cdot \nabla \mathbf{u}_2.
\]
Multiplying by \( \mu \), integrating over \( \Omega \), and using that \((u_1 \cdot \nabla \mu, \mu) = (1/2)(u_1, \nabla |\mu|^2)\), gives
\[
\frac{1}{2} \frac{d}{dt} \|\mu\|^2 + \frac{1}{2} \int_{\Omega} u_1 \cdot \nabla |\mu|^2 = -(w \cdot \nabla \omega_2, \mu) + (\omega_1 \cdot \nabla w, \mu) + (\mu \cdot \nabla u_2, \mu)
\]
\[
\leq \frac{1}{2} \|\nabla \omega_2\|_{L^\infty} \|w\|^2 + \frac{1}{2} \|\mu\|^2 + \frac{1}{2} \|\omega_1\|_{L^\infty} \|\nabla w\|^2 + \frac{1}{2} \|\mu\|^2 + \|\nabla u_2\|_{L^\infty} \|\mu\|^2,
\]
where \( \|\cdot\| := \|\cdot\|_{L^2(\Omega)} \) here. By Poincaré’s inequality, \( \|w\| \leq C \|\nabla w\| \), and since \( w \in H_0 \), we have \( \|\nabla w\| \leq C \|\mu\| \) and so obtain
\[
\frac{d}{dt} \|\mu\|^2 \leq -\int_{\Omega} u_1 \cdot \nabla |\mu|^2 + C \|\mu\|^2.
\]
We note that \( \nabla \omega_2 \in L^\infty([0, T] \times \Omega) \) by the \( N = 1 \) existence result. But,
\[
-\int_{\Omega} u_1 \cdot \nabla |\mu|^2 = \int_{\Omega} \text{div} u_1 |\mu|^2 - \int_{\Gamma} U^n |\mu|^2 = -\int_{\Gamma_-} U^n |\mu|^2 \leq 0,
\]
so we conclude from Gronwall’s lemma, since \( \mu(0) = 0 \), that \( \mu \equiv 0 \). That is, \( u_1 = u_2 \).

Finally, from (1.16), we have
\[
\partial_t u^T + (u \cdot \nabla u)^T = (f - \nabla p)^T + z^T.
\]
From cond_0, then, we see that \( z^T(0) = 0 \). Since also \( z^n(0) = 0 \), we know that \( z(0) = 0 \).

15. Impermeable boundary condition

If \( \Gamma_0 = \Gamma \), the classical setting of impermeable boundary conditions on the whole boundary, our proof of existence and uniqueness still applies, though a number of things trivialize. First, no vorticity is transported off of the boundary, so there is no need for the pressure estimates in Section 10, and \( U_- \) is all of \( Q \), so many of the flow map constructs, such as \( S \), \( \tau \), and \( \gamma \) are unnecessary. And, of course, none of the estimates involving \( U_+ \) are needed. The bound on the time of existence is still finite, however.

Appendix A. Hölder space lemmas

We collect here a number of bounds on products, compositions, and differences of functions in Hölder spaces, which we use throughout much of this paper. We include some proofs for the reader’s convenience.

Lemma A.1. Let \( f, g \in C^\alpha(U) \). Then
\[
\|fg\|_{C^\alpha} \leq \|f\|_{C^\alpha} \|g\|_{C^\alpha},
\]
\[
\|fg\|_{C_\alpha} \leq \|f\|_{L^\infty} \|g\|_{C_\alpha} + \|g\|_{L^\infty} \|f\|_{C_\alpha},
\]
\[
\|fg\|_{C^\alpha} \leq \|f\|_{L^\infty} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{C_\alpha} + \|g\|_{L^\infty} \|f\|_{C_\alpha},
\]
\[
\|fg\|_{C^\alpha} \leq \|f\|_{C^\alpha} \|g\|_{C_\alpha} + \|g\|_{L^\infty} \|f\|_{C^\alpha},
\]
\[
\|fg\|_{C^\alpha} \leq \|f\|_{L^\infty} \|g\|_{C_\alpha} + \|g\|_{L^\infty} \|f\|_{C^\alpha}.
\]

Also, for any \( \beta \in (0, \alpha) \), allowing \( \alpha = 1 \), we have the interpolation inequality,
\[
\|f\|_{C^\beta} \leq 2\|f\|_C^{\frac{\alpha}{\alpha - \beta}} \|f\|_{L^\infty}^{1 - \frac{\beta}{\alpha}}.
\]
Proof. These are all classical; we prove only the final inequality, which is perhaps less commonly encountered. We have,

\[ \|f\|_{\tilde{C}^\beta} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\beta}} \leq \sup_{x \neq y} \left[ \frac{|f(x) - f(y)|}{|x - y|^{\beta}} \right] \sup_{x \neq y} |f(x) - f(y)|^{1 - \frac{\beta}{\beta}} \leq 2 \|f\|_{\tilde{C}^\beta} \|f\|_{L^\infty}. \]

\[ \square \]

**Lemma A.2.** Let \( U, V \) be open subsets of Euclidean spaces, \( \alpha \in (0, 1] \), and \( k \geq 1 \) an integer. If \( f \in C^{k,\alpha}(U) \) and \( g \in C^{k,\alpha}(V) \) with \( g(V) \subseteq U \) then

\[
\|f \circ g\|_{C^\alpha(V)} \leq \|f\|_{C^\alpha(U)} \|g\|_{\text{Lip}(V)},
\]

\[
\|f \circ g\|_{C^\alpha(V)} \leq \|f\|_{L^\infty(U)} + \|f\|_{\tilde{C}^\alpha(U)} \|g\|_{\text{Lip}(V)} \leq \|f\|_{C^\alpha(U)} \left[ 1 + \|g\|_{C^{k+1}(V)} \right]^{k+1},
\]

where \( \text{Lip} \) is the homogeneous Lipschitz semi-norm and \( \tilde{C}^\alpha \) is the homogeneous Hölder norm.

Proof. Let \( x \neq y \). If \( g(x) \neq g(y) \) then

\[
\left| \frac{f(g(x)) - f(g(y))}{|x - y|^{\alpha}} \right| = \frac{|f(g(x)) - f(g(y))|}{|g(x) - g(y)|^\alpha} \left( \frac{|g(x) - g(y)|}{|x - y|} \right)^\alpha \leq \|f\|_{C^\alpha(U)} \|g\|_{\text{Lip}(V)}. 
\]

This gives (A.1)\(_1\). Combined with \( \|f \circ g\| \leq \|f\|_{L^\infty} \) we obtain (A.1)\(_2\).

For \( k = 1 \), we have

\[
\|D(f \circ g)\|_{C^\alpha(V)} \leq \|(Df \circ g)Dg\|_{C^\alpha(V)} \leq \|Df\|_{C^\alpha(U)} \|Dg\|_{C^\alpha(V)} \leq \|Df\|_{C^{1,\alpha}(U)} \|Dg\|_{C^{1,\alpha}(V)} \leq \|f\|_{C^{1,\alpha}(U)} \left[ 1 + \|g\|_{C^{1,\alpha}(V)} \right]^{1+\alpha},
\]

allowing for \( \alpha = 1 \).

\[ \square \]

**Lemma A.3.** Let \( U, V \) be open subsets of \( \mathbb{R}^d \), \( d \geq 1 \), and let \( \alpha \in (0, 1] \). Assume that the domain of \( f \) is \( U \) and the domains of \( g \) and \( h \) are \( V \), with \( g(V), h(V) \subseteq U \). Then

\[
\|f \circ g - f \circ h\|_{L^\infty(V)} \leq \|f\|_{C^\alpha(U)} \|g - h\|_{L^\infty(V)},
\]

Proof. For any \( x, y \in V \),

\[
|(f \circ g - f \circ h)(x)| = \left| \frac{f(g(x)) - f(h(x))}{g(x) - h(x)^\alpha} \right| g(x) - h(x)^\alpha \leq \|f\|_{C^\alpha(U)} \|g - h\|_{L^\infty(V)}. \]

We also have the following interpolation-like inequality:

**Lemma A.4.** Let \( U \) be a bounded open subset of \( \mathbb{R}^d \), \( d \geq 1 \), let \( n \geq 1 \), and \( \nabla^n f \in C^\alpha(U) \). Then

\[
\|\nabla^n f\|_{L^\infty(U)} \leq C \|f\|_{C^{\alpha,n}(U)} \|f\|_{L^\infty(U)}^{1-a},
\]

where

\[
a = a_n = \frac{2n + d}{2n + d + 2\alpha} < 1.
\]
Proof. First extend \( f \) continuously to all of \( \mathbb{R}^d \) in all Sobolov and Hölder spaces, as can be done using the extension operator in Theorem 5', chapter VI of [31]. Applying a cutoff function, we can insure that the extension, which we continue to call \( f \), has support with a diameter no more than twice \( \text{diam}(U) \).

Then
\[
\|\nabla^n f\|_{L^\infty(U)} = \sup_{x \in \text{supp} f} |\nabla^n f(x)| = \sup_{x \in \text{supp} f} |\nabla^n f(x) - \nabla^n f(x_0)| \leq R,
\]
where \( x_0 \) is a fixed point in \((\text{supp} f)^C\) and
\[
R = \sup_{x \in \text{supp} f} |x - x_0|^\alpha \sup_{x \in \text{supp} f} |\nabla^n f(x) - \nabla^n f(x_0)| = \sup_{x \in \text{supp} f} |x - x_0|^\alpha \|\nabla^n (f(s \cdot))\|_{C^\alpha(\mathbb{R}^d)}.
\]
In particular,
\[
\|\nabla^n f\|_{L^\infty(\mathbb{R}^d)} \leq R + \|f\|_{L^2(\mathbb{R}^d)}  \tag{A.2}
\]
for all \( f \in C_0^\infty(\mathbb{R}^d) \).

We have nearly the same setup as in the proof of Proposition 13.3.4 of [33]. Following the scaling argument in that proof, we write (A.2) schematically in the form \( Q \leq R + P \). Replacing \( f(\cdot) \) with \( f(s \cdot) \), we have \( \nabla^n(f(sx)) = s^n \nabla f(sx) \). This gives that \( \|\nabla^n (f(s \cdot))\|_{L^\infty(\mathbb{R}^d)} = s^n \|\nabla f\|_{L^\infty(\mathbb{R}^d)} \) and \( \|f(s \cdot)\|_{L^2(\mathbb{R}^d)} = s^{-\frac{d}{2}} \|f\|_{L^2(\mathbb{R}^d)} \). Also, \( R \) becomes
\[
\sup_{x \in \text{supp} f} |sx - sx_0|\alpha \sup_{x \in \text{supp} f} s^n |\nabla^n (f(sx) - \nabla^n f(sx_0))| = s^{n+\alpha} R.
\]
Thus, \( Q \leq R + P \) becomes
\[
s^n Q \leq s^{n+\alpha} R + s^{-\frac{d}{2}} P \implies Q \leq s^\alpha R + s^{-(n+\frac{d}{2})} P.
\]
As in [33], we conclude that
\[
\|\nabla^n f\|_{L^\infty(\mathbb{R}^d)} \leq \|\nabla^n f\|_{C^\alpha(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)}^{1-a} \leq C\|\nabla^n f\|_{C^\alpha(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)}^{1-a}
\]
as long as \( aa = (n + 1 - \frac{d}{2})(1 - a) \), which gives the stated value of \( a \) and the stated estimate, using the continuity of the extension operator. \( \square \)

The inequality in Lemma A.4 is very much like the lemma on page 126 of [27], which the authors of [3] use (for \( N = 0 \)) in place of it.

Lemma A.5. Let \( U \) be a bounded open subset of \( \mathbb{R}^d \), \( d \geq 1 \), let \( n \geq 1 \), and suppose that \( f \in C^{n,\alpha}(U) \). Let \( a_n \) be as in Lemma A.4. For any \( \beta \in (0, \alpha) \),
\[
\|f\|_{C^{n,\beta}(U)} \leq C\left[\|f\|_{C^{n,\alpha}(U)}^{\alpha_n} + \|f\|_{C^{n,\alpha}(U)}^{\frac{a_n}{\alpha_n}}\right] \left[\|f\|_{L^2(U)}^{1-\alpha_n} + \|f\|_{L^2(U)}^{1-a_n}\right]
+ C\|f\|_{C^{n,\alpha}(U)}^{a'} \|f\|_{L^2(U)}^{1-a'},
\]
where
\[
a' = (\beta/\alpha) + a_n(1 - \beta/\alpha) < 1.
\]
Proof. For any \( 1 \leq m \leq n \), Lemma A.4 gives
\[
\|\nabla^m f\|_{L^\infty(U)} \leq C\|f\|_{C^{n,\alpha}(U)}^{\frac{a_m}{\alpha_n}} \|f\|_{L^2(U)}^{1-\alpha_n}.
\]
Summed over \( m \) and combined with the trivial bound for \( m = 0 \) gives all but the final term in our bound on \( \|f\|_{C^{n,\beta}(U)} \) (we use that \( a_m \) is monotonically increasing in \( m \)). For that final
term, we apply the interpolation inequality in Lemma A.1 followed by Lemma A.4 to $\nabla^n f$, giving
\[
\|\nabla^n f\|_{C^\alpha(U)} \leq 2\|\nabla^n f\|_{C^\alpha(U)}^\alpha \leq C\|f\|_{C^\alpha(U)}^\alpha \|f\|_{L^\infty(U)}^{1-\alpha}.
\]

On $[0, T] \times \Gamma_+$, we have the following equivalent (using the regularity of $\Gamma_+$) formulations of Hölder norms (the analogous equivalent norms hold for any time-space domain, such as $Q$ and $U_\pm$):
\[
\|f\|_{C^\alpha([0,T] \times \Gamma_+)} := \sup_{(t,x) \neq (t',x')} \frac{|f(t,x) - f(t',x')|}{|t-t'|^\alpha} = \sup_{x \in \Gamma_+} \|f(x)\|_{C^\alpha([0,T])},
\]
\[
\|f\|_{C^\beta([0,T] \times \Gamma_+)} := \sup_{(t,x) \neq (t',x')} \frac{|f(t,x) - f(t',x')|}{|x-x'|^\beta} = \sup_{t \in [0,T]} \|f(t)\|_{C^\beta(\Gamma_+)},
\]
\[
\|f\|_{C^\gamma([0,T] \times \Gamma_+)} := \|f\|_{C^\gamma([0,T] \times \Gamma_+)} + \|f\|_{C^\gamma([0,T] \times \Gamma_+)}.
\]

Lemma A.6. Let $\alpha \in (0, 1]$ and assume that $f : [0, T] \times \Gamma_+ \to \mathbb{R}$ is a continuous function with the properties that for all $t_1, t_2 \in [0, T]$

(a) $\|f(t_1) - f(t_2)\|_{C^\alpha(\Gamma_+)} \leq F_1(|t_1 - t_2|);$  
(b) $\|f(t_1) - f(t_2)\|_{L^\infty(\Gamma_+)} \leq F_2(|t_1 - t_2|),$

where $F_1, F_2$ are increasing continuous functions with $F_2(t) = O(t^\alpha)$. Then
\[
\|f\|_{C^\alpha([0,T] \times \Gamma_+)} \leq \|f(0)\|_{C^\alpha(\Gamma_+)} + F_1(T) + \sup_{t \in [0,T]} \frac{F_2(t)}{t^\alpha}.
\]

Proof. First observe that
\[
\|f\|_{C^\gamma([0,T] \times \Gamma_+)} \leq \|f(0)\|_{C^\gamma(\Gamma_+)} + \|f(t) - f(0)\|_{C^\gamma(\Gamma_+)} \leq F_1(T).
\]

And by (A.3)$_2$,
\[
\|f(t) - f(0)\|_{C^\gamma(\Gamma_+)} \leq \sup_{t \in [0,T]} \|f(t) - f(0)\|_{C^\gamma(\Gamma_+)} \leq F_1(T).
\]

The bound on $\|f\|_{C^\gamma([0,T] \times \Gamma_+)}$ then follows from (A.3)$_3$.  

Lemma A.7. Assume that $f \in C^N,\alpha([0, T] \times \Gamma_+)$ for some $N \geq 0$, with the properties that for all $t_1, t_2 \in [0, T]$,

(a) $\|D^N f(t_1) - D^N f(t_2)\|_{C^\alpha(\Gamma_+)} \leq F_1(|t_1 - t_2|);$  
(b) $\|D^N f(t_1) - D^N f(t_2)\|_{L^\infty(\Gamma_+)} \leq F_2(|t_1 - t_2|),$

where $F_1, F_2$ are increasing continuous functions with $F_2(t) = O(t^\alpha)$. Then
\[
\|f\|_{C^N,\alpha([0,T] \times \Gamma_+)} \leq \|f(0)\|_{C^N,\alpha(\Gamma_+)} + \|f(t) - f(0)\|_{C^N([0,T] \times \Gamma_+)} + C F_1(T) + C \sup_{t \in [0,T]} \frac{F_2(t)}{t^\alpha}.
\]
Proof. We have,
\[
\|f\|_{C^{N,\alpha}(\Omega)} \leq \|f(0)\|_{C^{N,\alpha}(\Omega)} + \|f(t) - f(0)\|_{C^{N,\alpha}(\Omega)}.
\]
Applying Lemma A.6 gives the result. \qed

Lemma A.8. If \(f \in C^{N,\alpha}(Q)\) for some \(N \geq 0\) and \(\alpha \in (0,1]\) then for any \(t_1, t_2 \in [0,T]\),
\[
\|f(t_1) - f(t_2)\|_{C^{N}(Q)} \leq C\|f\|_{C^{N,\alpha}(Q)}|t_1 - t_2|^\alpha.
\]
Proof. We have,
\[
\|f(t_1) - f(t_2)\|_{C^{N}(Q)} = \sum_{|\gamma| < N} \|\partial_t^\gamma f\|_{L^\infty(Q)}|t_1 - t_2| + \|D^N f\|_{L^\infty(Q)}|t_1 - t_2|^\alpha
\]\[\leq C\|f\|_{C^{N,\alpha}(Q)}|t_1 - t_2|^\alpha,
\]
recalling Remark 6.2. \qed

Corollary A.9. If \(f \in C^{N,\alpha}(Q)\) for some \(N \geq 0\) and \(\alpha \in (0,1]\) then
\[
\|f(t) - f(0)\|_{C^{N}(Q)} \leq C\|f\|_{C^{N,\alpha}(Q)}T^\alpha.
\]
Proof. For any \(k \in (0,N)\) and \(t \in [0,T]\), applying Lemma A.8, we have
\[
\|\partial_t^k(f(t) - f(0))\|_{C^{N-k}(Q)} \leq C\|f\|_{C^{N-k,\alpha}(Q)}|t|^\alpha \leq C\|f\|_{C^{N,\alpha}(Q)}T^\alpha,
\]
from which the bound follows. \qed

Lemma A.10 is adapted from Lemma 8.3 of [18].

Lemma A.10. Suppose that \(f_j: \mathbb{R}^d \to \mathbb{R}, j = 1,2\), each have the modulus of continuity \(\mu\),
with \(\mu: [0,\infty) \to [0,\infty)\) continuous and increasing with \(\mu(0) = 0\). There exists a continuous
increasing function \(F: [0,\infty) \to [0,\infty)\), depending on \(\mu\), with \(F(0) = 0\) for which
\[
\|f_1 - f_2\|_{L^\infty(\mathbb{R}^d)} \leq F(\|f_1 - f_2\|_{L^2(\mathbb{R}^d)}).
\]
Proof. Fix \(x \in \mathbb{R}^d\) arbitrarily and suppose that \(\delta = |f_1(x) - f_2(x)| > 0\). Let \(y\) be in the ball
\(B\) of radius \(a = \mu^{-1}(\delta/4)\) about \(x\), so that \(|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)| \leq \delta/4\). Then
\[
|f_1(y) - f_2(y)| \geq \delta - |f_1(x) - f_1(y)| - |f_2(x) - f_2(y)| = \frac{\delta}{2}.
\]
Hence,
\[
\|f_1 - f_2\|_{L^2(\mathbb{R}^d)} \geq \|f_1 - f_2\|_{L^2(B)} \geq \left(\int_B \left(\frac{\delta}{2}\right)^2 \right)^{1/2} = \frac{\delta}{2} \sqrt{\pi a},
\]
or,
\[
h(\delta) := \frac{\sqrt{\pi}}{2} \mu^{-1}(\delta/4) \leq \|f_1 - f_2\|_{L^2(\mathbb{R}^d)}.
\]
Since \(\mu^{-1}\) must be increasing, so must \(h\), so setting \(F = h^{-1}\) (noting that \(F(0) = 0\)) we have
\[
|f_1(x) - f_2(x)| = \delta \leq F(\|f_1 - f_2\|_{L^2(\mathbb{R}^d)}).
\]
This inequality applies for all \(x\) even when \(\delta = |f_1(x) - f_2(x)| = 0\), giving the result. \qed
APPENDIX B. BOUNDARY CALCULATIONS

In this appendix we give explicit expressions for $\nabla$, div, and curl using curvilinear coordinates in a fixed $\varepsilon$-neighborhood $\Sigma$ of $\Gamma$ of arbitrary width $\delta > 0$. We cannot work exclusively on the boundary, for we need to take derivatives in the normal direction; but in application we always evaluate our expressions on the boundary itself, which will simplify the expressions somewhat and lead to our definitions of $\nabla\Gamma$, $\text{div}\Gamma$, and $\text{curl}\Gamma$ on the boundary, which we used in the body of the paper. These formulas are classical in differential geometry. We include a derivation for completeness, following Section 3.1.1 of [9].

In 3D, we say a point on $\Gamma$ is an \textit{umbilical point} if its principal curvatures, $\kappa_1$ and $\kappa_2$, are equal. Let $x_0$ be a non-umbilical point. It is always possible to construct a coordinate system $(\xi_1, \xi_2)$ on $\Gamma$ in some neighborhood of $x_0$ for which the coordinate lines are parallel to the principal directions (directions giving the principal curvatures, which are always orthogonal) and the surface metric (derived as an embedding in $\mathbb{R}^3$) is diagonal at each point. This follows, for instance, from Lemma 3.6.6 of [19]. Supplementing this coordinate system with $\xi_3 = \text{signed distance from the boundary to a point}$, with negative values inside the domain, we have a coordinate system $(\xi_1, \xi_2, \xi_3)$ locally on some $\Sigma' \subseteq \Sigma$. Such coordinates are called \textit{principal curvature coordinates}. They are orthogonal but not, unless the surface is locally flat, orthonormal. (Note $\xi_3 < 0$ in $\Sigma \cap \Omega$, so $+n$ is in the direction of increasing $\xi_3$.)

The issues involving umbilical points are discussed in some depth in Section 4.2 of [11], but we will find, as was found in [11], that once we use these coordinates to obtain our expressions on the boundary, they can be re-expressed in coordinate-free form as in Proposition 9.2, and will continue to hold even for umbilical points.

Writing $x = (x_1, x_2, x_3)$ for Cartesian coordinates, the boundary, a 2D surface, can be parameterized locally by a function $\gamma$:

$$x = \gamma(\xi_1, \xi_2) = (\gamma^1(\xi_1, \xi_2), \gamma^2(\xi_1, \xi_2), \gamma^3(\xi_1, \xi_2)).$$  \hfill (B.1)

Then we define (unnormalized) coordinate vectors,

$$g_j := \frac{\partial \gamma}{\partial \xi_j}, \quad j = 1, 2$$

on the boundary. In the frame $(g_1, g_2)$, we write the diagonal metric tensor in the form $\text{diag}(a_1, a_2)$ with $a_1, a_2 > 0$.

Adding a third vector, $g_3 := n$, the outward-directed normal vector,

$$x = \gamma(\xi_1, \xi_2) + \xi_3 n$$ \hfill (B.2)

is the mapping from $\xi$-coordinates to Cartesian coordinates, and we obtain a diagonal metric tensor in $\Sigma'$ of the form,

$$g = \begin{pmatrix}
(1 + \kappa_j \xi_3)^2 a_1^2 & 0 & 0 \\
0 & (1 + \kappa_2 \xi_3)^2 a_2^2 & 0 \\
0 & 0 & 1
\end{pmatrix}.$$

Observe that $1 + \kappa_j \xi_3 > 0$, $j = 1, 2$, in $\Sigma'$, since it is an $\varepsilon$-neighborhood.

Let $h_i := \sqrt{|g_{ii}|}$, $h := \sqrt{\det g}$, so

$$h_1 = (1 + \kappa_1 \xi_3)a_1, \quad h_2 = (1 + \kappa_2 \xi_3)a_2, \quad h_3 = 1, \quad h = h_1 h_2.$$  \hfill (B.3)

By the definition of the metric tensor,

$$|g_j| = (1 + \kappa_j \xi_3) a_j = h_j.$$  \hfill (B.3)
We now introduce normalized coordinated vectors,

\[ \tau_1 := \frac{g_1}{h_1} = \frac{\mathbf{e}_1}{h_1}, \quad \tau_2 := \frac{g_2}{h_2} = \frac{\mathbf{e}_2}{h_2}, \quad \tau_3 := \mathbf{n}. \]

Then \((\tau_1, \tau_2, \mathbf{n})\) are orthonormal and point in the direction of the coordinate lines. As we can see from (B.2), \(\tau_1, \tau_2\) are constant along lines normal to the boundary, that is for fixed \((\xi_1, \xi_2)\). (The vectors \(g_1, g_2\) vary with \(\xi_3\), but only in magnitude.)

Moreover, by switching \(\xi_1\) and \(\xi_2\) if necessary, we can insure that \((\tau_1, \tau_2)\) is in the standard orientation of \((\mathbf{e}_1, \mathbf{e}_2)\) on the boundary (looking from “outside” the domain) while at the same time \((\tau_1, \tau_2, \mathbf{n})\) is in the standard orientation of \((\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\).

It follows from (3.8), (3.12), and (3.13) of [9] that in our coordinates (where the convention of implicit sums over repeated indices applies to the tangential components, \(j = 1, 2\), only)

\[ \nabla f = \frac{1}{h_j} \frac{\partial f}{\partial \xi_j} \tau_j + \frac{1}{h_3} \frac{\partial f}{\partial \xi_3} \mathbf{n} = \frac{1}{(1 + \kappa_1 \xi_3) a_j} \frac{\partial f}{\partial \xi_j} \tau_j + \frac{\partial f}{\partial \xi_3} \mathbf{n} \]

for a scalar function \(f\). For

\[ \mathbf{v} = v^k \tau_k = v^1 \tau_1 + v^2 \tau_2 + v^3 \mathbf{n}, \]

we have

\[ \text{div} \mathbf{v} = \frac{1}{h} \frac{\partial}{\partial \xi_j} \left( \frac{h_j}{h} v^j \right) + \frac{1}{h} \frac{\partial}{\partial \xi_3} (hv^3) \]

and

\[ \text{curl} \mathbf{v} = \frac{h_1}{h} \left[ \frac{\partial v^3}{\partial \xi_2} - \frac{\partial (hv^2)}{\partial \xi_3} \right] \tau_1 + \frac{h_2}{h} \left[ \frac{\partial (h v^1)}{\partial \xi_3} - \frac{\partial v^3}{\partial \xi_1} \right] \tau_2 + \frac{h_1}{h} \left[ \frac{\partial (hv^2)}{\partial \xi_1} - \frac{\partial (h v^1)}{\partial \xi_2} \right] \mathbf{n}. \]

On the boundary,

\[ h_1 = a_1, \quad h_2 = a_2, \quad h_3 = 1, \quad h = a_1 a_2, \quad \frac{h}{h_1} = a_2, \quad \frac{h}{h_2} = a_1, \]

\[ \frac{\partial h_1}{\partial \xi_1} = \frac{\partial a_1}{\partial \xi_1}, \quad \frac{\partial h_2}{\partial \xi_1} = \frac{\partial a_2}{\partial \xi_1}, \quad \frac{\partial h_1}{\partial \xi_3} = \kappa_1 a_1, \quad \frac{\partial h_2}{\partial \xi_3} = \kappa_2 a_2, \]

\[ \frac{1}{h} \frac{\partial h}{\partial \xi_1} = \frac{1}{a_1} \frac{\partial h_1}{\partial \xi_1} = \frac{1}{a_2} \frac{\partial h_2}{\partial \xi_1} = \kappa_1 + \kappa_2. \]

The gradient therefore reduces very simply to

\[ \nabla f = \frac{1}{a_1} \frac{\partial f}{\partial \xi_1} \tau_1 + \frac{1}{a_2} \frac{\partial f}{\partial \xi_2} \tau_2 + \frac{\partial f}{\partial \xi_3} \mathbf{n}. \]

For the divergence, we have

\[ \text{div} \mathbf{v} = \frac{1}{a_1 a_2} \left[ \frac{\partial (a_2 v^1)}{\partial \xi_1} + \frac{\partial (a_1 v^2)}{\partial \xi_2} \right] + \frac{\partial v^3}{\partial \xi_3} + (\kappa_1 + \kappa_2) v^3. \]

Finally, for the curl,

\[ \text{curl} \mathbf{v} = \frac{1}{a_2} \left[ \frac{\partial v^3}{\partial \xi_2} - \frac{\partial (hv^2)}{\partial \xi_3} \right] \tau_1 + \frac{1}{a_1} \left[ \frac{\partial (h v^1)}{\partial \xi_3} - \frac{\partial v^3}{\partial \xi_1} \right] \tau_2 + \frac{1}{a_1 a_2} \left[ \frac{\partial (hv^2)}{\partial \xi_1} - \frac{\partial (h v^1)}{\partial \xi_2} \right] \mathbf{n} \]

\[ + \frac{1}{a_1 a_2} \left[ \frac{\partial h}{\partial \xi_1} - \frac{\partial (h v^1)}{\partial \xi_2} \right] \mathbf{n}. \]
\[
= \left[ \frac{1}{a_2} \frac{\partial v^3}{\partial \xi_2} - \frac{\partial v^2}{\partial \xi_3} \right] \tau_1 + \left[ \frac{\partial v^1}{\partial \xi_3} - \frac{1}{a_1} \frac{\partial v^3}{\partial \xi_1} \right] \tau_2 - \kappa_2 v^2 \tau_1 + \kappa_1 v^1 \tau_2
\]

\[
+ \frac{1}{a_1 a_2} \left[ \frac{\partial (h_2 v^2)}{\partial \xi_1} - \frac{\partial (h_1 v^1)}{\partial \xi_2} \right] n.
\]

To summarize, on \( \Gamma \),
\[
\nabla f = \frac{1}{a_1} \frac{\partial f}{\partial \xi_1} \tau_1 + \frac{1}{a_2} \frac{\partial f}{\partial \xi_2} \tau_2 + \frac{\partial f}{\partial \xi_3} n,
\]
\[
\mathrm{div} \ v = \frac{1}{a_1 a_2} \left[ \frac{\partial (a_2 v^1)}{\partial \xi_1} + \frac{\partial (a_1 v^2)}{\partial \xi_2} \right] + \frac{\partial v^3}{\partial \xi_3} + (\kappa_1 + \kappa_2) v^3,
\]
\[
curl \ v = \left[ \frac{1}{a_2} \frac{\partial v^3}{\partial \xi_2} - \frac{\partial v^2}{\partial \xi_3} \right] \tau_1 + \left[ \frac{\partial v^1}{\partial \xi_3} - \frac{1}{a_1} \frac{\partial v^3}{\partial \xi_1} \right] \tau_2 - \kappa_2 v^2 \tau_1 + \kappa_1 v^1 \tau_2
\]

\[
+ \frac{1}{a_1 a_2} \left[ \frac{\partial (h_2 v^2)}{\partial \xi_1} - \frac{\partial (h_1 v^1)}{\partial \xi_2} \right] n.
\]

Locally our coordinates define a collection of \( M \) charts, which we use to cover the \( \varepsilon \)-neighborhood \( \Sigma \). For simplicity, we will confine ourselves to writing expressions in a single chart, however.

Define \( \nabla_\Gamma \), \( \mathrm{div}_\Gamma \) to be the gradient and divergence operators on \( \Gamma \). We can define them in a coordinate free manner by requiring that for any \( f \in C^\infty(\Gamma) \) and any smooth curve \( x(s) \) on \( \Gamma \) parameterized by arc length,
\[
\nabla_\Gamma f \cdot x'(0) = \lim_{s \to 0} \frac{f(x(s)) - f(x(0))}{s}.
\]

(B.5)

It is easy to verify that locally in principal curvature coordinates,
\[
\nabla_\Gamma f = \frac{1}{a_1} \frac{\partial f}{\partial \xi_1} \tau_1 + \frac{1}{a_2} \frac{\partial f}{\partial \xi_2} \tau_2.
\]

(B.6)

Lemma B.1. Let \( f \in C^1(\Gamma) \), \( v \in (C^1(\Gamma))^d \). Then
\[
\int_{\Gamma} v \cdot \nabla_\Gamma f = -\int_{\Gamma} \mathrm{div}_\Gamma v \ f,
\]
where \( \mathrm{div}_\Gamma \) can be written in our coordinates as
\[
\mathrm{div}_\Gamma v = \frac{1}{a_1 a_2} \left[ \frac{\partial (a_2 v^1)}{\partial \xi_1} + \frac{\partial (a_1 v^2)}{\partial \xi_2} \right].
\]

(B.7)

Moreover,
\[
\mathrm{div}_\Gamma (f v) = f \mathrm{div}_\Gamma v + v \cdot \nabla_\Gamma f.
\]

(B.8)

Proof. This is classical for smooth functions (see, for instance, Proposition 2.2.2 of [32]), and follows in the same way for \( C^1 \) functions, integrating by parts on the boundary in charts. \( \square \)

We collect now a few useful facts.

For \( u, v \) tangent vectors,
\[
(u \cdot \nabla_\Gamma v) \cdot v = \frac{1}{a_j} u^j \partial_j |v|^2 = \frac{1}{2 a_j} u^j \partial_j |v|^2 = \frac{1}{2} u \cdot \nabla |v|^2,
\]
so for any component \( \Gamma_n \) of the boundary,
\[
\int_{\Gamma_n} (u \cdot \nabla_\Gamma v) \cdot v = \frac{1}{2} \int_{\Gamma_n} u \cdot \nabla_\Gamma |v|^2.
\]
We also define (with the ⊥ operator as in Definition 9.1)
\[
\text{curl}_\Gamma \mathbf{v} := -\text{div}_\Gamma \mathbf{v}^\perp = \frac{1}{a_1 a_2} \left[ \frac{\partial (a_2 v^2)}{\partial \xi_1} - \frac{\partial (a_1 v^1)}{\partial \xi_2} \right] = (\text{curl} \mathbf{v}) \cdot \mathbf{n}.
\] (B.9)

Observe that \text{div}_\Gamma \mathbf{v} depends only upon the value of \mathbf{v} on \Gamma, and that
\[
\int_{\Gamma_j} \text{curl}_\Gamma \mathbf{v} = 0
\] (B.10)
for each boundary component \( \Gamma_j \) of \( \Gamma \) follows from Lemma B.1. Also,
\[
\text{curl}_\Gamma \nabla f = \text{curl}_\Gamma \left[ \frac{1}{a_1} \frac{\partial f}{\partial \xi_1} \tau_1 + \frac{1}{a_2} \frac{\partial f}{\partial \xi_2} \tau_2 \right] = \frac{1}{a_1 a_2} \left[ \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2} - \frac{\partial^2 f}{\partial \xi_2 \partial \xi_1} \right] = 0.
\] (B.11)

**Lemma B.2.** Let \( \mathbf{u}, \mathbf{v} \) be vector fields on \( \overline{\Omega} \). Then
\[
[\mathbf{u} \times \mathbf{v}]^\tau = u^n[\mathbf{v}^\tau]^\perp - v^n[\mathbf{u}^\tau]^\perp.
\]

**Proof.** We have,
\[
\mathbf{u} \times \mathbf{v} = (\mathbf{u}^n + \mathbf{u}^\tau) \times (\mathbf{v}^n + \mathbf{v}^\tau) = \mathbf{u}^n \times \mathbf{v}^\tau - \mathbf{v}^n \times \mathbf{u}^\tau + \mathbf{u}^\tau \times \mathbf{v}^\tau,
\]
since \( \mathbf{u}^n \times \mathbf{v}^n = 0 \). Now, \( \mathbf{u}^\tau \times \mathbf{v}^\tau \) is parallel to \( \mathbf{n} \), so we see that
\[
[\mathbf{u} \times \mathbf{v}]^\tau = \mathbf{u}^n \times \mathbf{v}^\tau - \mathbf{v}^n \times \mathbf{u}^\tau.
\]
But, \( \mathbf{u}^n \) is perpendicular to \( \mathbf{v}^\tau \), so we see that \( \mathbf{u}^n \times \mathbf{v}^\tau = u^n[\mathbf{v}^\tau]^\perp \), and similarly, \( \mathbf{v}^n \times \mathbf{u}^\tau = v^n[\mathbf{u}^\tau]^\perp \). Hence, \( [\mathbf{u} \times \mathbf{v}]^\tau = u^n[\mathbf{v}^\tau]^\perp - v^n[\mathbf{u}^\tau]^\perp \). \( \square \)

We will need one final identity, which follows from (B.4):
\[
\text{div} \mathbf{v} = \text{div}_\Gamma \mathbf{v}^\tau + \frac{\partial v^3}{\partial \xi_3} + (\kappa_1 + \kappa_2) v^3 \text{ on } \Gamma.
\] (B.12)

**Proof of Proposition 9.2.** Because \((\tau_1, \tau_2, \tau_3)\) is orthonormal on \( \Gamma \), if \( \mathbf{u} = u^1 \tau_1 + u^2 \tau_2 + v^3 \tau_3 \) and \( \mathbf{v} = v^1 \tau_1 + v^2 \tau_2 + v^3 \tau_3 \), recalling that \( \tau_3 = \mathbf{n} \), then \( \mathbf{u} \cdot \mathbf{v} = u^j v^j \). And because we are evaluating expressions only on the boundary, we can use (B.4). Defining
\[
b_1 := \frac{1}{a_1}, \quad b_2 := \frac{1}{b_2}, \quad b_3 = 1,
\]
we see, then, that \((\mathbf{u} \cdot \nabla \mathbf{u})^i = b^j u^j \partial_j u^i \), so
\[
(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{n} = b^j u^j \partial_j u_n + u_n b^j \partial_j (u^j n^i) - u^j b^j \partial_j (u^j n^i) = \mathbf{u} \cdot \nabla (\mathbf{u} \cdot \mathbf{n}) - \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{n}).
\]

Decomposing \( \mathbf{u} = u^n + \mathbf{u}^\tau \), we write the first term as
\[
\mathbf{u} \cdot \nabla (\mathbf{u} \cdot \mathbf{n}) = (u^n \mathbf{n} + \mathbf{u}^\tau) \cdot (\partial_n u^n + \nabla_\Gamma u^n) = u^n \partial_n u^n + \mathbf{u}^\tau \cdot \nabla_\Gamma u^n.
\]

But \( \text{div} \mathbf{u} = 0 \), so (B.12) gives
\[
\partial_n u^n = -\text{div}_\Gamma \mathbf{u}^\tau - (\kappa_1 + \kappa_2) u^n.
\]

Hence, we can write,
\[
\mathbf{u} \cdot \nabla (\mathbf{u} \cdot \mathbf{n}) = \mathbf{u}^\tau \cdot \nabla_\Gamma u^n - u^n \text{div}_\Gamma \mathbf{u}^\tau - (\kappa_1 + \kappa_2) u^n.
\]

Now, \( \mathbf{u}^\tau \cdot \nabla \mathbf{n} = \mathcal{A} \mathbf{u}^\tau \), where \( \mathcal{A} \) is the shape operator. Also, \( \mathbf{u}^n \cdot \nabla \mathbf{n} = u^n \partial_n \mathbf{n} = 0 \). Thus, since \( \mathcal{A} \mathbf{u}^\tau \) lie in the tangent plane,
\[
\mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{n}) = \mathbf{u}^\tau \cdot \mathcal{A} \mathbf{u}^\tau.
\]
We conclude that
\[(u \cdot \nabla u) \cdot n = u^T \cdot \nabla_{\Gamma} u^n - u^n \div_{\Gamma} u^T - (\kappa_1 + \kappa_2)(u^n)^2 - u^T \cdot A u^T.\]

Alternately, we can use that \(\div_{\Gamma}(u^n u^T) = u^T \cdot \nabla_{\Gamma} u^n + u^n \div_{\Gamma} u^T\) to write
\[(u \cdot \nabla u) \cdot n = 2u^T \cdot \nabla_{\Gamma} u^n - \div_{\Gamma}(u^n u^T) - (\kappa_1 + \kappa_2)(u^n)^2 - u^T \cdot A u^T.\]

This gives (9.1). (We can write \(u^T \cdot A u^T = \kappa_1 u^1 + \kappa_2 u^2\) by the comment following Lemma 3.6.6 of [19], but \(u^T \cdot A u^T\) is the coordinate-free form.) \(\square\)

**Appendix C. Compatibility conditions: special case**

In [34], Temam and Wang consider a periodic domain with \(\mathcal{U} = (0, 0, -1)\), so \(\mathcal{U}^T = 0\) for all time. More generally, the authors of [9] consider \(\mathcal{U} = -U^I n\), where \(U^I > 0\) is constant, so \(\mathcal{U}^T = 0\) on \(\Gamma_+\) for all time. The compatibility conditions simplify in these settings, as in Proposition C.1.

**Proposition C.1.** Assume that \(\mathcal{U}^T \equiv 0\) and \(U^n\) is spatially constant along \(\Gamma_+\) \((U^n\) need not be constant in time). The compatibility condition \(\text{cond}_N\) for \(N \geq 0\) is
\[
\partial_t^j f^T|_{t=0} = \partial_t^j \nabla_{\Gamma} p|_{t=0} - U^n_0 (\partial_t^j \omega^T)^\perp|_{t=0} \text{ for all } 0 \leq j \leq N, \tag{C.1}
\]
where \(\partial_t^j \nabla_{\Gamma} p|_{t=0}\) and \(\partial_t^j \omega|_{t=0}\) must be treated as explained following (1.15).

**Proof.** From (B.9), \(u^T = \mathcal{U}^T = 0\), which means that on \(\Gamma_+\),
\[
\omega^n = \omega \cdot n = \curl_{\Gamma} u^T = 0.
\]
In particular, this holds at time zero. Both \(\partial_t \mathcal{U}^T = 0\) and \(\curl_{\Gamma} \mathcal{U}^T = 0\), while \(|\mathcal{U}|^2 = (U^n)^2\) is constant on \(\Gamma_+\), so also \(\nabla_{\Gamma} |\mathcal{U}|^2 = 0\). We see, then, that \(H^T\) simplifies to \(H^T = (U^n)^{-1} [f^T - \nabla_{\Gamma} p]^\perp\), so \(\text{cond}_N\) becomes
\[
[f^T - \nabla_{\Gamma} p]_{t=0}^\perp = U^n_0 \omega^T_0,
\]
which is (C.1) for \(N = 0\). The inductive extension of this to higher \(N\) follows readily, leading to (C.1) for \(N \geq 0\). \(\square\)

The condition in (C.1) for \(N = 0\) also follows from \(\text{cond}_0\) with slightly more work, though the inductive extension to higher \(N\) is not so transparent as it is starting from \(\text{cond}_0\).

Because \(\div f = 0\) with \(f \cdot n = 0\) on \(\Gamma\), \(f\) plays no role in the calculation of \(\nabla_{\Gamma} p\) for \(N = 0\). By writing the condition in (C.1) with the forcing on one side, and the quantities coming from the initial data on the other, we are stressing that, given initial data one can always choose a forcing at time zero so that \(\text{cond}_0\) is satisfied.

For all \(N \geq 1\), though, forcing enters into the calculation of \(\partial_t \nabla_{\Gamma} p\) through replacing \(\partial_t u_0\) with \(f - u_0 \cdot \nabla u_0 - \nabla p_0\): even though \(f \cdot n = 0\), the forcing still does not, in general, vanish from even the \(N = 1\) condition. Because of this, the forcing is intimately entwined in \(\text{cond}_N\) for \(N \geq 1\), appearing on both sides of the condition, even for the simplest nontrivial case considered in [34]. These same comments hold in the general setting, they are just perhaps clearer here where the conditions simplify somewhat.
References


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