Vanishing Viscosity Limits for a Class of Circular Pipe Flows *

Anna Mazzucato
Department of Mathematics, Penn State University
University Park, PA 16802, USA

Michael Taylor
Department of Mathematics, University of North Carolina
Chapel Hill, NC 27599, USA

Abstract

We consider 3D Navier-Stokes flows with no-slip boundary condition in an infinitely long pipe with circular cross section. The velocity fields we consider are independent of the variable parametrizing the axis of the pipe, and the component of the velocity normal to the axis is arranged to be circularly symmetric, though we impose no such symmetry on the component of velocity parallel to the axis. For such flows we analyze the limit as the viscosity tends to zero, including boundary layer estimates.

1 Introduction

In this paper we study a class of solutions to the 3D Navier-Stokes equations

$$\frac{\partial u^\nu}{\partial t} + \nabla u^\nu u^\nu + \nabla p^\nu = \nu \Delta u^\nu + F^\nu, \quad \text{div} \ u^\nu = 0,$$

(1.1)

for $u^\nu = u^\nu(t, x, z)$, $p^\nu = p^\nu(t, x, z)$ with $(t, x, z) \in \mathbb{R}^+ \times \Omega$, where

$$\Omega = D \times \mathbb{R}, \quad D = \{x \in \mathbb{R}^2 : |x| < 1\}.$$  

(1.2)

We denote the closure of $D$ by $\overline{D}$, with boundary $\partial D$. We restrict attention to the following type of external force field $F^\nu$:

$$F^\nu(t, x, z) = (0, f^\nu(t)),$$

(1.3)

i.e., $F^\nu$ is parallel to the $z$-axis, with $z$-component $f^\nu(t)$. We impose no-slip boundary data on the boundary, which might be rotating and translating:

$$u^\nu(t, x, z) = \left(\frac{\alpha(t)}{2\pi} x^\perp, \beta(t)\right), \quad |x| = 1, \ z \in \mathbb{R}, \ t > 0.$$  

(1.4)

Here $x^\perp = Jx$ where $J$ is counterclockwise rotation by $90^\circ$. We take initial data of the following form:

$$u^\nu(0, x, z) = u_0(x) = (v_0(x), w_0(x)),$$

(1.5)

*2000 Math Subject Classification. 35Q30, 35K20, 35B25

Key words: Navier-Stokes equations, viscosity, boundary layer, singular perturbation
The first author was partially supported by NSF grant DMS-0708902.
where \( v_0 \) is a vector field on \( D \) and \( w_0 \) is the \( z \)-component of \( u_0 \). We require the conditions
\[
\text{div } u_0 = 0, \quad u_0 \parallel \partial \Omega, \quad \text{i.e.,} \quad \text{div } v_0 = 0, \quad v_0 \parallel \partial D, \quad (1.6)
\]
and we require that the vector field \( v_0 \) on \( D \) be circularly symmetric.

By definition, a vector field \( v_0 \) on \( D \) is circularly symmetric provided
\[
v_0(R_\theta x) = R_\theta v_0(x), \quad \forall x \in D, \quad (1.7)
\]
for each \( \theta \in [0, 2\pi] \), where \( R_\theta \) is counterclockwise rotation by \( \theta \). The general planar vector field satisfying (1.7) has the form \( v_0(|x|)x^\perp + s_1(|x|)x \), with \( s_j \) scalar, but the condition \( \text{div } v_0 = 0 \), together with the condition \( v_0 \parallel \partial D \), forces \( s_1 \equiv 0 \), so the type of initial data we consider is characterized by
\[
u_0(x) = (s_0(|x|)x^\perp, w_0(x)). \quad (1.8)
\]
Another characterization of this special form for \( v_0 \) is that
\[
v_0(\Phi_\omega x) = -\Phi_\omega v_0(x) \quad (1.9)
\]
for all \( \omega \in S^1 \), where \( \Phi_\omega : \mathbb{R}^2 \to \mathbb{R}^2 \) is the reflection across the line generated by \( \omega \).

The fact that \( \Omega \) in (1.2) is infinite makes uniqueness an issue. (We discuss this further in Appendix B.) To guarantee uniqueness, we modify the set-up by requiring the solutions to be periodic (say of period \( L \)) in \( z \), i.e., we replace \( \Omega \) in (1.2) by
\[
\Omega_L = D \times (\mathbb{R}/L\mathbb{Z}). \quad (1.10)
\]
In such a case, the general theory implies that (1.1), (1.4) and (1.5) has a unique strong, short-time solution, given mild regularity hypotheses on \( v_0(x) \) and \( w_0(x) \) (actually the solution persists globally in \( t \), as we will see shortly) and the solution is \( z \)-translation invariant, i.e.,
\[
u(t,x) = (v(t,x), w(t,x)), \quad p(t,x). \quad (1.11)
\]

**Remark.** While \( F' \), given by (1.3), satisfies
\[
F'(t,x) = \nabla f'(t)z, \quad \text{this is not the gradient of a function periodic in } z.
\]

Note that
\[
\nabla_{v'} v' = (\nabla_{v'} v', \nabla_{v'} w'), \quad \text{div } v' = \text{div } v'. \quad (1.12)
\]
Hence, in the current setting, (1.1) is equivalent to the following system of equations on \( \mathbb{R}^+ \times D \):
\[
\frac{\partial v'}{\partial t} + \nabla_{v'} v' + \nabla p' = \nu \Delta v', \quad \text{div } v' = 0, \quad (1.13)
\]
\[
\frac{\partial w'}{\partial t} + \nabla_{v'} w' = \nu \Delta w' + f'(t). \quad (1.14)
\]
Note that (1.13) is the 2D Navier-Stokes equation for flow on \( D \). We are imposing the boundary condition
\[
v'(t,x) = \frac{\alpha(t)}{2\pi} x^\perp, \quad |x| = 1, \quad t > 0, \quad (1.15)
\]
and the initial condition
\[
v'(0,x) = v_0(x) = s_0(|x|)x^\perp. \quad (1.16)
\]
Meanwhile, (1.14) is a scalar equation, with boundary condition

\[ w^\nu(t, x) = \beta(t), \quad |x| = 1, \quad t > 0, \] (1.17)

and initial condition

\[ w^\nu(0, x) = w_0(x). \] (1.18)

We do not require \( v_0(x) \) to equal \( \frac{\alpha(0)}{2\pi} x^\perp \) when \( x \in \partial D \), nor do we require \( w_0(x) \) to equal \( \beta(0) \) when \( x \in \partial D \). At this point we recall that the solvability of (1.13) for all \( t \in \mathbb{R}^+ \), for each \( \nu > 0 \), is well known, and the solvability of (1.14) for all \( t \in \mathbb{R}^+ \) is then relatively elementary.

Our main goal is to study the limit as \( \nu \downarrow 0 \) of the solutions to (1.13)–(1.18), and see how \( u^\nu = (v^\nu, w^\nu) \) approaches the solution of the Euler equation

\[
\frac{\partial u^0}{\partial t} + \nabla \cdot u^0 + \nabla p^0 = F^0, \quad \text{div} \ u^0 = 0,
\] (1.19)

on \( \mathbb{R}^+ \times \Omega_L \), with initial condition

\[ u^0(0, x, z) = (v_0(x), w_0(x)), \] (1.20)

given in (1.16) and (1.18), and with boundary condition

\[ u^0 \| \partial \Omega_L. \] (1.21)

Here \( F^0(t, x, z) = (0, f^0(t)) \). Arguments as above give

\[ u^0(t, x, z) = (v^0(t, x), w^0(t, x)), \] (1.22)

where \( v^0(t, x) \) and \( w^0(t, x) \) solve

\[
\frac{\partial v^0}{\partial t} + \nabla \cdot v^0 + \nabla p^0 = 0, \quad \text{div} \ v^0 = 0, \] (1.23)

\[
\frac{\partial w^0}{\partial t} + \nabla \cdot w^0 = f^0(t). \] (1.24)

Note that (1.23) is the 2D Euler equation for flows on \( D \). We have the boundary condition

\[ v^0(t, x) \| \partial D \quad \text{for} \quad t > 0, \quad x \in \partial D, \] (1.25)

and initial condition

\[ v^0(0, x) = v_0(x) = s_0(|x|) x^\perp. \] (1.26)

As is well known, the vector field \( v_0 \) given by (1.26) is a steady solution to the Euler equation (1.23). In fact, a calculation gives

\[ \nabla v_0 \cdot v_0 = -s_0(|x|)^2 x = -\nabla p_0(x), \] (1.27)

with

\[ p_0(x) = \tilde{p}_0(|x|), \quad \tilde{p}_0(r) = -\int_r^1 \rho s_0(\rho)^2 \, d\rho, \] (1.28)

which proves our assertion:

\[ v^0(t, x) \equiv v_0(x), \] (1.29)
when $v_0(x)$ is as in (1.26). From here, we see that (1.24) becomes
\[
\frac{\partial w_0}{\partial t} + \nabla v_0 w_0 = f^0(t).
\] (1.30)

The tangency condition (1.21) imposes no boundary condition for $w_0$. This is logical, since $\partial_t + \nabla v_0$ is a vector field on $\mathbb{R} \times D$ that is tangent to $\mathbb{R} \times \partial D$. The solution to (1.30), with initial condition
\[
w_0(0, x) = w_0(x),
\] (1.31)
is given by
\[
w_0(t, x) = w_0(F^{-t}_v(x)) + \int_0^t f^0(s) \, ds,
\] (1.32)
where $F^{-t}_v : D \rightarrow D$ is the backwards flow on $D$ generated by $v_0$.

Now the task of analyzing how $u^\nu \rightarrow u^0$ as $\nu \searrow 0$ has two parts, namely how
\[
v^\nu \rightarrow v^0 \quad \text{as } \nu \searrow 0,
\] (1.33)
and how
\[
w^\nu \rightarrow w^0 \quad \text{as } \nu \searrow 0.
\] (1.34)

There is a literature on (1.33), including [7], [10], [1], and, recently, [5] and [6]. The first key to a successful attack on (1.33) is the following result.

**Proposition 1.1** Given that $v_0$ has the form (1.26), the solution $v^\nu$ to (1.13), (1.15), (1.16) is circularly symmetric for each $t > 0$, of the form
\[
v^\nu(t, x) = s^\nu(t, |x|) x^\perp,
\] (1.35)
and it coincides with the solution to the linear PDE
\[
\frac{\partial v^\nu}{\partial t} = \nu \Delta v^\nu,
\] (1.36)
with boundary condition (1.15) and initial condition (1.16).

This well known result figured in the analyses in the papers cited above. A proof (using the characterization (1.9)) is recorded in Proposition 1.1 of [6]. We mention in particular that
\[
\nabla v^\nu \cdot v^\nu = -\nabla p^\nu, \quad p^\nu(t, x) = \tilde{p}^\nu(t, |x|),
\]
\[
\tilde{p}^\nu(t, r) = -\int_r^1 \rho s^\nu(t, \rho)^2 \, d\rho.
\] (1.37)

The structure of the rest of this paper is as follows. In §2 we recall results of [5] and [6] on the nature of the convergence $v^\nu \rightarrow v^0$ in (1.33), and give some further results, which will be of use in §3. Prior results include a variety of $L^p$-Sobolev space estimates, recalled in Propositions 2.3–2.5. Further results include estimates in spaces $V^k(D)$ (defined in (2.33)), available thanks to [8], given in Proposition 2.6 and Corollary 2.7. New results (of crucial use in §3) include explicit boundary layer analyses, following from material in Appendix A, leading to estimates in the space $V^{\infty, \infty}(D)$ (defined in (2.52)), given in Propositions 2.8–2.10.

In §3 we discuss the nature of the convergence $w^\nu \rightarrow w^0$ in (1.34). Here we apply results obtained in [8]. These results were originally directed towards a different fluid problem, involving
plane parallel channel flows, but [8] found it convenient to develop the relevant singular perturbation theory on a more general level, and, thanks to the results of §2 of this paper, this development has applications to (1.34). In (3.21) we write

\[ w^\nu(t, x) - w^0(t, x) = R_1(\nu, t, x) + R_2(\nu, t, x) + R_3(\nu, t, x), \]  

(1.38)

and apply a variety of attacks on the three terms on the right, which are defined in (3.18)–(3.20). We obtain estimates on \( R_2 \) and \( R_3 \) in \( L^p(D) \), for \( p \in [1, \infty) \) in Proposition 3.1, and such estimates on \( R_1 \) in Proposition 3.3. These results lead to \( w^\nu(t, \cdot) \to w^0(t, \cdot) \) in such \( L^p \)-norms. Propositions 3.6–3.8 yield \( w^\nu(t, \cdot) \to w^0(t, \cdot) \) in the spaces \( V^k(D) \), leading to convergence boundedly and locally uniformly on \( D \), established in Proposition 3.9.

Of course, Proposition 3.9 does not establish uniform convergence on \( \overline{D} \). There is a boundary layer effect at work here, as there was for \( v^\nu \to v \). The corresponding study of boundary layer effects for \( w^\nu \to w \) is taken up in §4. Again we use the decomposition (1.38) and apply separate analyses to the three terms on the right. Results of Appendix A give an explicit boundary layer analysis of \( R_2 \). Layer potential techniques developed in [8] give an almost equally precise analysis of \( R_1 \). This leaves \( R_3 \), and as we show in (4.47)–(4.51), we can obtain at least an estimate on boundary layer thickness for this term, consistent with the boundary layer thickness results apparent for \( R_1 \) and \( R_2 \), though fine detail on the boundary layer behavior of \( R_3 \) remains a topic for further work.

This paper ends with two appendices. Appendix A, which has already been mentioned, studies limits as \( \nu \searrow 0 \) of solutions to

\[ \frac{\partial u^\nu}{\partial t} = \nu \Delta u^\nu \quad \text{on} \quad \mathbb{R}^+ \times \Omega, \]  

(1.39)

satisfying

\[ u^\nu \big|_{\mathbb{R}^+ \times \partial \Omega} = 0, \quad u^\nu(0, x) = f(x). \]  

(1.40)

We take \( f \in C^\infty(\overline{\Omega}) \), and do not require it to vanish on the boundary. We make use of wave equation techniques to produce explicit boundary layer analyses of solutions to this equation, whose utility is manifested in §§2–4. Appendix B discusses a class of Poisseuille flows, and places their analysis in the context of problems treated in this paper.

As indicated above, this paper is a companion to [6] and [8]. In fact, it combines features of these two papers, yielding a family of flows somewhat more complex than either the 2D case of [6] or the plane-parallel channel flows of [8]. The circular pipe flows considered here are not only subtler than the latter case, but also, it seems, of greater intuitive appeal. When producing [8], the authors performed much of the analysis on a fairly general level, hoping for more generally applicable results. Nevertheless, we did not realize at the time that these general results could be made to apply to pipe flow problems. The tools that make this work are presented in the latter part of §2 and in Appendix A.

2 Nature of the convergence \( v^\nu \to v^0 \)

As explained in the introduction, the component of the solution to (1.1)–(1.8) normal to the axis of the pipe solves the linear system

\[ \frac{\partial v^\nu}{\partial t} = \nu \Delta v^\nu \quad \text{on} \quad \mathbb{R}^+ \times D, \]  

(2.1)

\[ v^\nu(t, x) = \frac{\alpha(t)}{2\pi} x^\perp \quad \text{on} \quad (0, \infty) \times \partial D, \]  

(2.2)

\[ v^\nu(0, x) = v_0(x) = s_0(|x|) x^\perp. \]  

(2.3)
Here, for each \( t \geq 0 \), \( v^\nu(t, \cdot) \) is a planar vector field on the disk \( \overline{D} = \{ x \in \mathbb{R}^2 : |x| \leq 1 \} \), tangent to the boundary. We do not require \( s_0(1) \) to be equal to \( \alpha(0)/2\pi \). This non-matching is what produces the boundary layer effect. In this section we recall some results from [5] and [6] on the nature of the convergence \( v^\nu \to v^0 \equiv v_0 \), and produce some additional results, which will be of use in §3.

One tool to analyze solutions to (2.1)–(2.3) is the semigroup \( e^{t\Delta} \), defined by 
\[
\frac{\partial u}{\partial t} = \Delta u \quad \text{on} \quad \mathbb{R}^+ \times D, \quad u|_{\mathbb{R}^+ \times \partial D} = 0, \quad u(0) = f. \tag{2.4}
\]
We have
\[
v^\nu(t) = e^{\nu t \Delta} v_0 + \mathcal{S}^\nu \alpha, \tag{2.5}
\]
where \( \mathcal{S}^\nu \alpha = V^\nu \) solves
\[
\frac{\partial V^\nu}{\partial t} = \nu \Delta V^\nu, \quad V^\nu = 0 \quad \text{for} \quad t < 0,
\]
\[
V^\nu|_{\mathbb{R}^+ \times \partial D} = \frac{\alpha(t)}{2\pi} x^\perp. \quad \tag{2.6}
\]
If we set
\[
C_b^\infty(\mathbb{R}) = \{ \alpha \in C^\infty(\mathbb{R}) : \alpha(t) = 0 \quad \text{for} \quad t < 0 \},
\]
\[
C_b(\mathbb{R}) = \{ \alpha \in C(\mathbb{R}) : \alpha(t) = 0 \quad \text{for} \quad t < 0 \}, \tag{2.7}
\]
we have for each \( \nu > 0 \),
\[
\mathcal{S}^\nu : C_b^\infty(\mathbb{R}) \longrightarrow C_b^\infty(\mathbb{R} \times \overline{D}), \quad \mathcal{S}^\nu : C_b(\mathbb{R}) \longrightarrow C_b(\mathbb{R} \times \overline{D}), \tag{2.8}
\]
where the subscript \( b \) in the spaces on the right side of (2.8) also denote vanishing for \( t < 0 \), as it does in (2.9) below. As shown in [6], we have a continuous extension
\[
\mathcal{S}^\nu : L^p_b(\mathbb{R}) \longrightarrow C([0,1], H_{\text{loc},b}^{-1}(\mathbb{R} \times \partial D)), \tag{2.9}
\]
where we use polar coordinates \([0,1] \times \partial D \rightarrow \overline{D}, \ (r, e^{i\theta}) \mapsto re^{i\theta}\). In each case (2.8)–(2.9),
\[
\text{Tr}(\mathcal{S}^\nu \alpha) = \frac{\alpha}{2\pi} x^\perp, \quad \tag{2.10}
\]
for each \( \nu > 0 \).

The behavior as \( \nu \searrow 0 \) of the first term of the right side of (2.5) is governed by the behavior as \( t \searrow 0 \) of \( e^{t\Delta} \), acting on \( u_0 \). The behavior of \( \mathcal{S}^\nu \alpha \) as \( \nu \searrow 0 \) is attacked as follows. First assume \( \alpha \in C_b^\infty(\mathbb{R}) \). Set
\[
\tilde{V}^\nu(t, x) = V^\nu(t, x) - \frac{\alpha(t)}{2\pi} x^\perp. \tag{2.11}
\]
This solves
\[
\frac{\partial \tilde{V}^\nu}{\partial t} = \nu \Delta \tilde{V}^\nu - \alpha'(t) f_1, \quad \tilde{V}^\nu(0) = 0, \quad \tilde{V}^\nu|_{\mathbb{R}^+ \times \partial D} = 0, \quad \tag{2.12}
\]
where
\[
f_1(x) = \frac{x^\perp}{2\pi}. \tag{2.13}
\]
Hence, by Duhamel’s formula,
\[
\tilde{V}^\nu(t) = - \int_0^t e^{\nu(t-s)\Delta} f_1 \alpha'(s) \, ds. \tag{2.14}
\]
Substitution into (2.11) gives
\[ S^\nu \alpha(t) = \int_0^t (I - e^{\nu(t-s)\Delta}) f_1 \alpha'(s) \, ds. \] (2.15)

A mollifier argument gives the following (Proposition 2.1 of [6]).

**Proposition 2.1** Let \( X \) be a Banach space of functions on \( D \) such that \( f_1 \in X \) and \( \{e^{t\Delta} : t \geq 0\} \) is a strongly continuous semigroup on \( X \). Then
\[ S^\nu : BV_b(\mathbb{R}) \to C_b(\mathbb{R}, X), \] (2.16)

with
\[ S^\nu \alpha(t) = \int_{I(t)} (I - e^{\nu(t-s)\Delta}) f_1 \alpha(s) \, ds, \] (2.17)

where we can take either \( I(t) = [0, t] \) or \( I(t) = [0, t) \). Furthermore,
\[ S^\nu \alpha(t) = -\lim_{\varepsilon \to 0} \nu \int_0^{t-\varepsilon} e^{\nu(t-s)\Delta} f_1 \alpha(s) \, ds. \] (2.18)

**Corollary 2.2** In the setting of Proposition 2.1,
\[ \|S^\nu \alpha(t)\|_X \leq \|\alpha\|_{BV([0, t])} \sup_{s \in [0, t]} \|e^{\nu s \Delta} f_1 - f_1\|_X. \] (2.19)

Hence, if \( \nu^\nu \) solves (2.1)-(2.3) and \( \nu_0 \in X \), then
\[ \|\nu^\nu(t) - \nu_0\|_X \leq \|e^{\nu t \Delta} \nu_0 - \nu_0\|_X + \|\alpha\|_{BV([0, t])} \sup_{s \in [0, t]} \|e^{\nu s \Delta} f_1 - f_1\|_X. \] (2.20)

The following records spaces \( X \) to which Proposition 2.1 applies.

**Proposition 2.3** \( \{e^{t\Delta} : t \geq 0\} \) is a strongly continuous semigroup on the following spaces:
\[ L^p(D), \quad 1 \leq p < \infty, \] (2.21)
more generally the \( L^p \)-Sobolev spaces
\[ H^{s,p}(D), \quad 1 \leq p < \infty, \quad 0 \leq s < \frac{1}{p}, \] (2.22)

Also,
\[ C_*(D) = \{f \in C(\overline{D}) : f|_{\partial D} = 0\}, \] (2.23)
\[ H^1_0(D) = \{f \in H^{1,2}(D) : f|_{\partial D} = 0\}, \] (2.24)
and
\[ H^\sigma_0(D) \cap H^{\sigma,2}(D), \quad 1 \leq \sigma < \frac{5}{2}. \] (2.25)

7
See [6] for more details and references. We mention that \( \{e^{t\Delta} : t \geq 0\} \) is a contraction semigroup on the spaces (2.21), and also on
\[
L^\infty(D), \quad C(\overline{D}),
\]
but it is not strongly continuous at \( t = 0 \) on the spaces (2.26).

Proposition 2.3 has obvious applications to the limiting behavior as \( \nu \searrow 0 \) of the first term on the right side of (2.5). As for \( S'\alpha \), Proposition 2.3 together with the formulas (2.17) and (2.18) can be used to establish the following (Proposition 4.2 of [6]).

**Proposition 2.4** Assume \( q \in (1, \infty) \) and assume
\[
0 \leq \sigma < \tau < \frac{1}{q}, \quad p \in \left[1, \frac{2}{2 - 1/q + \sigma}\right).
\]
Then
\[
S'^\nu : L^{p'}_b(\mathbb{R}) \longrightarrow C_b(\mathbb{R}, H^{\sigma,q}(D)),
\]
and
\[
\|S'^\nu \alpha(t)\|_{H^{\sigma,q}(D)} \leq C(t) \nu^{(\tau - \sigma)/2} \|\alpha\|_{L^{p'}([0,t])} \|f_1\|_{H^{\tau,q}(D)},
\]
provided that also
\[
1 \leq p < \frac{2}{2 - (\tau - \sigma)}.
\]

Note. For a given \( p \), there exist \( q, \tau, \sigma \) satisfying the hypotheses above, provided \( 1 \leq p < 2 \), i.e., provided \( p' > 2 \).

In addition to such global convergence results as given above, there are local convergence results, which hold in stronger norms, such as the following (Proposition 7.1 of [6]).

**Proposition 2.5** Let \( \mathcal{O} \subset D \) be open, \( \Omega \subset \overline{\Omega} \subset \mathcal{O} \), \( \Omega \) smoothly bounded. Assume
\[
v_0 \in L^2(D), \quad v_0|_{\mathcal{O}} \in H^k(\mathcal{O}), \quad \alpha \in L^1_b(\mathbb{R}).
\]
Then, given \( T_0 < \infty \),
\[
\lim_{\nu \searrow 0} v^\nu(t)|_{\Omega} = v_0|_{\Omega} \quad \text{in} \quad H^k(\Omega),
\]
uniformly for \( t \in [0, T_0] \).

Proposition 2.5 applies in particular when \( v_0 \in H^k(D) \). In such a case, we can draw a stronger conclusion, via some analysis done in [8], which will also prove useful in §3. We introduce the following spaces:
\[
\mathcal{V}^k(D) = \{u \in L^2(D) : Lu \in L^2(D), \forall L \in \mathcal{X}^k\},
\]
where
\[
\mathcal{X}^k = \text{Span} \{Z_1 \cdots Z_j : j \leq k, \ Z_\ell \in \mathcal{X}^1\},
\]
with
\[
\mathcal{X}^1 = \{Y \text{ smooth vector field on } \overline{D} : Y \parallel \partial D\}.
\]
We note that there exists a finite family \( \{Y_j : 1 \leq j \leq M\} \subset \mathcal{X}^1 \) that spans \( \mathcal{X}^1 \) over \( C^\infty(\overline{D}) \). In fact, we can take \( M = 3 \) and
\[
Y_j = (1 - r^2) \frac{\partial}{\partial x_j} \quad (j = 1, 2), \quad Y_3 = \frac{\partial}{\partial \theta}.
\]
We can set
\[ Y^J = Y_{j_1} \cdots Y_{j_k}, \quad |J| = k, \quad (2.37) \]
and
\[ \|u\|_{V_k}^2 = \sum_{|J| \leq k} \|Y^J u\|_{L^2}^2. \quad (2.38) \]

We mention that, for each \( k \in \mathbb{Z}^+ \),
\[ C_0^\infty(D) \text{ is dense in } V_k(D). \quad (2.39) \]
The following result is a special case of results of §3.3 of [8], to which we will return in the following section of this paper.

**Proposition 2.6** For each \( k \in \mathbb{Z}^+ \), \( \{e^{t\Delta} : t \geq 0\} \) is a strongly continuous semigroup on \( V_k(D) \).

We can then bring in Corollary 2.2 and deduce:

**Corollary 2.7** If \( \nu \) solves (2.1)–(2.3) and \( v_0 \in V_k(D) \), then
\[ \|v_\nu(t) - v_0\|_{V_k} \leq \|e^{t\Delta} v_0 - v_0\|_{V_k} + \|\alpha\|_{BV([0,t])} \sup_{s \in [0,t]} \|e^{\nu s \Delta} f_1 - f_1\|_{V_k}, \quad (2.40) \]
which tends to 0 as \( \nu \searrow 0 \), uniformly for \( t \in [0,T_0] \), provided \( \alpha \in BV([0,T_0]) \).

We now describe more detailed behavior of \( e^{t\Delta} v_0(x) \) in case \( v_0 \in C^\infty(D) \). Parallel to Proposition 2.5, we have the interior regularity result
\[ v_0 \in C^\infty(D), \quad v(t,x) = e^{t\Delta} v_0(x) \implies v \in C^\infty([0,\infty) \times D), \quad (2.41) \]
as well as \( v \in C^\infty((0,\infty) \times \overline{D}) \). It remains to analyze the behavior near \( t = 0 \) on a neighborhood of \( \partial D \). In [6] this was attacked via the use of layer potentials. In Appendix A of this paper we use another method, exploiting a connection with the wave equation and the method of geometrical optics. In Proposition A.1 we exhibit \( e^{t\Delta} v_0(x) \) near \( \partial D \) (for \( v_0 \in C^\infty(D) \)) as
\[ e^{t\Delta} v_0(x) = v_0(x) + \sum_{k=1}^N \frac{\ell_k}{k!} \Delta^k v_0(x) \]
\[ - \sum_{j=0}^{2N} 2b_j(x)(4t)^{j/2} E_j\left(\frac{\varphi(x)}{\sqrt{4t}}\right) + \hat{R}_N(t,x). \quad (2.42) \]

Here we have
\[ b_j \in C^\infty(D), \quad \varphi(x) = 1 - |x|, \quad (2.43) \]
and
\[ E_j(y) = \frac{1}{\sqrt{\pi}} \int_y^\infty e^{-s^2} (s - y)^j ds \]
\[ = e^{-y^2} \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2 - 2sy} s^j ds. \quad (2.44) \]
The term \( \hat{R}_N \) is a remainder. Its significance is that, for each \( M, k \in \mathbb{N} \) there exists \( N \) such that
\[ \|\hat{R}_N(t,\cdot)\|_{C^k(\overline{D})} \leq C_M k^M, \quad t \in (0,1]. \quad (2.45) \]
Note that for each $j \geq 0$, $E_j \in C^\infty([0, \infty))$ is positive and rapidly decreasing at infinity. The boundary layer phenomenon is captured by these terms, particularly the leading term

$$-2b_0(x)E_0\left(\frac{1 - |x|}{\sqrt{4t}}\right).$$

(2.46)

Note that $E_0(0) = 1/2$, and hence

$$b_0(x) = v_0(x) \text{ for } x \in \partial D.$$  

(2.47)

These results apply to $f_1$, given by (2.13). In this case, $\Delta f_1 = 0$, and we get

$$e^{t\Delta}f_1(x) = f_1(x) - \sum_{j=0}^{2N} 2g_j(x)(4t)^{j/2}E_j\left(\frac{1 - |x|}{\sqrt{4t}}\right) + \hat{R}_N(t, x),$$

(2.48)

with $g_j \in C^\infty(\overline{D})$. By (2.17), we get

$$S^\nu_\alpha(t) = \sum_{j=0}^{2N} 2g_j(x) \int_0^t (4\nu(t - s))^{j/2}E_j\left(\frac{1 - |x|}{\sqrt{4\nu(t - s)}}\right) d\alpha(s)$$

(2.49)

$$- \int_0^t \hat{R}_N(\nu(t - s), x) d\alpha(s).$$

The estimate (2.45) implies

$$\left\| \int_0^t \hat{R}_N(\nu(t - s), \cdot) d\alpha(s) \right\|_{C^k(\overline{D})} \leq C_{M,k}(T)\|\alpha\|_{BV} \nu^M, \quad 0 < t \leq T.$$  

(2.50)

By (2.5), the results (2.42) (with $t$ replaced by $\nu t$) and (2.49) apply to produce an asymptotic expansion for $v^\nu(t, x)$ as $\nu \searrow 0$, valid uniformly for $t \in [0, T]$.

We use these asymptotic results to obtain further estimates on $e^{t\Delta}v_0$, for $v_0 \in C^\infty(\overline{D})$, which will be of use in §3. To set this up, we introduce the following generalization of $V_k(D)$ in (2.33).

Given $k \in \mathbb{Z^+}$, $p \in [1, \infty]$, set

$$V^{k,p}(D) = \{u \in L^p(D) : Lu \in L^p(D), \forall L \in \mathfrak{X}^k\}. $$

(2.51)

Also set

$$V^{\infty,p}(D) = \bigcap_k V^{k,p}(D). $$

(2.52)

**Proposition 2.8** Given $v_0 \in C^\infty(\overline{D})$, we have

$$\{e^{t\Delta}v_0 : t \geq 0\} \text{ bounded in } V^{\infty,\infty}(D). $$

(2.53)

**Proof.** The bound in $C^\infty(\overline{D}) \subset V^{\infty,\infty}(D)$ for $t \geq 1$ is elementary. Also the bound in $C^\infty(\{x : |x| \leq 1/2\})$ for $t \in [0, 1]$ follows from (2.41). To finish, it suffices to show that

$$\left\{E_j\left(\frac{1 - |x|}{\sqrt{4t}}\right) : 0 < t \leq 1\right\} \text{ is bounded in } V^{\infty,\infty},$$

(2.54)

near $\partial D$, which follows from the assertion that

$$\left\{(y \frac{d}{dy})^k E_j\left(\frac{y}{\sqrt{4t}}\right) : 0 < t \leq 1\right\} \text{ is bounded in } L^\infty([0, 1]), \forall j, k.$$  

(2.55)
The identity
\[ y \frac{d}{dy} E_j \left( \frac{y}{\sqrt{4t}} \right) = \frac{y}{\sqrt{4t}} E'_j \left( \frac{y}{\sqrt{4t}} \right) \]
gives (2.55) for \( k = 1 \), and the result for general \( k \) follows by induction.

Applying this to (2.5) and (2.17), we have:

**Proposition 2.9** Given \( v' \) solving (2.1)–(2.3) with \( v_0 \in C^\infty(D) \), \( \alpha \in BV_b(R) \), we have
\[ \{ v' : \nu \in (0, 1) \} \text{ bounded in } V^{\infty, \infty}(D), \] uniformly for \( t \in [0, T_0] \), \( T_0 < \infty \).

Recall from Proposition 1.1 that solutions to (2.1)–(2.3) have the form
\[ v' = s' \left( t, \frac{|x|}{2} \right), \]
The following complement to Proposition 2.9 provides the key to applying results from [8] to prove Proposition 3.7 in the next section.

**Proposition 2.10** When (2.55) and (2.56) hold, we have
\[ \{ s'(t, \cdot) : \nu \in (0, 1), t \in [0, T_0] \} \text{ bounded in } V^{\infty, \infty}(D). \]
We have
\[ \| \tilde{s}^{\nu}(t, \cdot) \|_{C^1([D_{1/2}])} \leq C \| \sigma^{\nu}(t, \cdot) \|_{C^2([-1/4,1/4])}. \]  
(2.67)

Clearly
\[ \| \sigma^{\nu}(t, \cdot) \|_{C^0([-1/4,1/4])} = \| s^{\nu}(t, \cdot) \|_{C^0([-1/2,1/2])}. \]  
(2.68)

Next
\[
\partial_{\rho} \sigma^{\nu}(t, \rho) = \frac{1}{2} \rho^{-1/2} \partial_{r} s^{\nu}(t, \rho^{1/2})
= \frac{1}{2} \int_{0}^{1} \partial_{r}^{2} s^{\nu}(t, \rho^{1/2} \tau) d\tau,
\]  
(2.69)

the latter identity using \( \partial_{r} s^{\nu}(t, 0) = 0 \). Inductively, we obtain
\[ \| \sigma^{\nu}(t, \cdot) \|_{C^1([-1/4,1/4])} \leq C \| s^{\nu}(t, \cdot) \|_{C^{2k}([-1/2,1/2])}. \]  
(2.70)

Recalling (2.65) and (2.67), we get
\[ \| \tilde{s}^{\nu}(t, \cdot) \|_{C^k([D_{1/2}])} \leq C \| s^{\nu}(t, \cdot) \|_{C^{2k}([-1/2,1/2])}
\leq C \| v^{\nu}(t, \cdot) \|_{C^{2k+1}([D_{1/2}])}, \]  
(2.71)

finishing the proof of Proposition 2.10. \( \square \)

We record some more estimates that follow from the arguments given above. Since
\[
\int_{D} |x|^{-p} dx = 2\pi \int_{0}^{1} r^{1-p} dr < \infty \text{ for } p \in [1,2),
\]  
(2.72)

we have
\[ \| \tilde{s}^{\nu}(t, \cdot) \|_{L^p(D)} \leq C \| v^{\nu}(t, \cdot) \|_{L^p(D)} + C \| v^{\nu}(t, \cdot) \|_{L^\infty(D_{1/2})}, \quad p \in [1,2). \]  
(2.73)

We also have
\[ \| \tilde{s}^{\nu}(t, \cdot) \|_{L^p(D)} \leq C \| v^{\nu}(t, \cdot) \|_{L^p(D)} + C \| v^{\nu}(t, \cdot) \|_{C^{1}([D_{1/2}])}, \quad p \in [1,\infty]. \]  
(2.74)

These estimates will be useful in the proof of Proposition 3.1.

### 3 Nature of the convergence \( w^{\nu} \to w^0 \)

Recall from the introduction that the component of the solution to (1.1)–(1.8) parallel to the axis of the pipe solves the scalar equation
\[
\frac{\partial w^{\nu}}{\partial t} = \nu \Delta w^{\nu} - X_{\nu} w^{\nu} + f^{\nu}(t) \quad \text{on } \mathbb{R}^+ \times D, \]  
(3.1)

\[ w^{\nu}(t, x) = \beta(t) \quad \text{on } (0, \infty) \times \partial D, \]  
(3.2)

\[ w^{\nu}(0, x) = w_0(x), \]  
(3.3)

where
\[ X_{\nu} g = \nabla_{\nu} g = s^{\nu}(t, |x|) \frac{\partial g}{\partial \theta}. \]  
(3.4)
We do not require \( w_0(x) = \beta(0) \) for \( x \in \partial D \). We assume \( v^\nu \) is given by (2.1)–(2.3), with
\[
v_0 \in C^\infty(\overline{D}), \quad \alpha \in BV_b(\mathbb{R}),
\]
so the results of §2 are available. In this section we show how results of §2 allow us to apply results of [8] to draw conclusions about the nature of the convergence \( w^\nu \rightarrow w^0 \), the solution to
\[
\frac{\partial w^0}{\partial t} = -Xw^0 + f^0(t), \quad w^0(0, x) = w_0(x),
\]
where
\[
Xg = \nabla_{v_0}g = s_0(|x|) \frac{\partial g}{\partial \theta}.
\]

The initial-boundary problem (3.1)–(3.3) is a singular perturbation of (3.6)–(3.8) in two ways, due both to the presence of \( \nu \Delta w^\nu \) in (3.1) and to the nature of the convergence of the coefficients of \( X^\nu \) to those of \( X \). We can partially separate these two mechanisms by rewriting (3.1) as
\[
\frac{\partial w^\nu}{\partial t} = (\nu \Delta - X)w^\nu + (X - X^\nu)w^\nu + f^\nu(t).
\]
Also let us set
\[
W^\nu(t, x) = w^\nu(t, x) - \beta(t),
\]
so \( W^\nu(t, x) \) solves
\[
\frac{\partial W^\nu}{\partial t} = (\nu \Delta - X)W^\nu + (X - X^\nu)W^\nu + g^\nu(t),
\]
\[
W^\nu(t, x) = 0 \quad \text{on} \quad (0, \infty) \times \partial D,
\]
\[
W^\nu(0, x) = W^0(x) = w_0(x) - \beta(0),
\]
where
\[
g^\nu(t) = f^\nu(t) - \beta'(t),
\]
assuming \( \beta \in C^1_b(\mathbb{R}) \). (We relax this requirement below.) Duhamel’s formula gives
\[
W^\nu(t) = e^{t(\nu \Delta - X)}W_0 + \int_0^t e^{(t-s)(\nu \Delta - X)}(XW^\nu - X^\nu W^\nu + g^\nu(s)) \, ds
\]
\[
= e^{t(\nu \Delta - X)}W_0 + \int_0^t e^{(t-s)(\nu \Delta - X)}(s_0 - s^\nu) \frac{\partial W^\nu}{\partial \theta} \, ds
\]
\[
+ \int_0^t g^\nu(s)e^{(t-s)(\nu \Delta - X)} \, ds.
\]

By comparison, if we set
\[
W^0(t, x) = w^0(t, x) - \beta(t),
\]
which solves
\[
\frac{\partial W^0}{\partial t} = -XW^0 + g^0(t), \quad W^0(0, x) = W_0(x),
\]
with
\[
g^0(t) = f^0(t) - \beta'(t),
\]
we have
\[
W^0(t) = e^{-tX}W_0 + \int_0^t g^0(s) \, ds.
\]
To compare \( w^\nu(t, x) \) and \( w^0(t, x) \), we will separately estimate
\[
R_1(\nu, t, x) = e^{t(\nu \Delta - X)}W_0 - e^{-tX}W_0, \quad (3.18)
\]
\[
R_2(\nu, t, x) = \int_0^t g^\nu(s)e^{(t-s)(\nu \Delta - X)}1 - g^0(s) \, ds, \quad \text{and} \quad (3.19)
\]
\[
R_3(\nu, t, x) = \int_0^t e^{(t-s)(\nu \Delta - X)}(s_0 - s^\nu) \frac{\partial W^\nu}{\partial \theta} \, ds, \quad (3.20)
\]
which fit together as follows:
\[
w^\nu(t, x) - w^0(t, x) = W^\nu(t, x) - W^0(t, x) = \sum_{j=1}^3 R_j(\nu, t, x). \quad (3.21)
\]

We begin with an estimate on \( R_3(\nu, t, x) \). Note that
\[
\frac{\partial}{\partial \theta} \text{ commutes with } X, X_\nu, \text{ and } \Delta. \quad (3.22)
\]
Hence
\[
Z^\nu = \frac{\partial W^\nu}{\partial \theta} \implies
\partial Z^\nu = (\nu \Delta - X_\nu)Z^\nu, \quad Z^\nu \big|_{\mathbb{R}^+ \times \partial D} = 0, \quad Z^\nu(0) = \frac{\partial W_0}{\partial \theta} = \frac{\partial w_0}{\partial \theta}. \quad (3.23)
\]
Thus the maximum principle gives
\[
\left\| \frac{\partial W^\nu}{\partial \theta} (s) \right\|_{L^\infty(D)} \leq \left\| \frac{\partial W_0}{\partial \theta} \right\|_{L^\infty(D)} = \left\| \frac{\partial w_0}{\partial \theta} \right\|_{L^\infty(D)}. \quad (3.24)
\]
Since the semigroup \( e^{t(\nu \Delta - X)} \) is positivity preserving, we have
\[
|R_3(\nu, t, x)| \leq \left\| \frac{\partial w_0}{\partial \theta} \right\|_{L^\infty} \int_0^t e^{(t-s)(\nu \Delta - X)}|s_0(x) - s^\nu(x)| \, ds. \quad (3.25)
\]
We also have, by radial symmetry,
\[
e^{(t-s)(\nu \Delta - X)}|s_0 - s^\nu| = e^{\nu(t-s)\Delta}|s_0 - s^\nu|, \quad (3.26)
\]
and hence
\[
|R_3(\nu, t, x)| \leq \left\| \frac{\partial w_0}{\partial \theta} \right\|_{L^\infty} \int_0^t e^{\nu(t-s)\Delta}|s_0 - s^\nu| \, ds. \quad (3.27)
\]
Here, as in (2.58), \( \tilde{s}^\nu(t, x) = s^\nu(t, |x|) \), and similarly \( \tilde{s}_0(x) = s_0(|x|) \).

Moving on to \( R_2(\nu, t, x) \), we have, as in (3.26),
\[
e^{(t-s)(\nu \Delta - X)}1 = e^{\nu(t-s)\Delta}1, \quad (3.28)
\]
and hence
\[
R_2(\nu, t, x) = \int_0^t \left[ (g^\nu(s) - g^0(s)) + g^\nu(s)(e^{\nu(t-s)\Delta}1 - 1) \right] \, ds
= \int_0^t [f^\nu(s) - f^0(s)] \, ds + \int_0^t g^\nu(s)(e^{\nu(t-s)\Delta}1 - 1) \, ds
= \int_0^t [f^\nu(s) - f^0(s)] \, ds + R_2^\#(\nu, t, x). \quad (3.29)
\]
Using (3.12), we can write
\[
R_2^\#(\nu, t, x) = \int_0^t f'(s)(e^{\nu(t-s)}1 - 1) \, ds + \int_0^t (e^{\nu(t-s)}1 - 1) \, d\beta(s),
\] (3.30)
and by a mollifier argument such as described in §1 extend the validity of this identity from \(\beta \in C^1_b(\mathbb{R})\) to \(\beta \in BV_b(\mathbb{R})\).

We record some \(L^p\)-estimates on \(R_3\) and \(R_2\). Since \(e^{t\Delta}\) is a contraction semigroup on \(L^p(D)\), (3.27) yields
\[
\|R_3(\nu, t, \cdot)\|_{L^p} \leq \|\partial_\nu \omega_0\|_{L^\infty} \int_0^t \|\tilde{s}_0 - \tilde{s}'(\cdot, \cdot)\|_{L^p} \, ds
\]
\[
\leq \|\partial_\nu \omega_0\|_{L^\infty} \sup_{s \in [0, t]} \|\tilde{s}_0(\cdot) - \tilde{s}'(\cdot, \cdot)\|_{L^p} \cdot t.
\] (3.31)

Meanwhile, (3.30) yields
\[
\|R_2^\#(\nu, t, \cdot)\|_{L^p} \leq \left(\|f'\|_{L^1([0, t])} + \|\beta\|_{BV([0, t])}\right) \sup_{s \in [0, t]} \|1 - e^{\nu(t-s)1}\|_{L^p}.
\] (3.32)

The arguments yielding (2.73)–(2.74) also yield
\[
\|\tilde{s}_0(\cdot) - \tilde{s}'(t, \cdot)\|_{L^p(D)}
\leq C\|\nu'(t, \cdot) - v_0(\cdot)\|_{L^p(D)} + C\|\nu'(t, \cdot) - v_0(\cdot)\|_{L^\infty(D_{1/2})}, \quad p \in [1, 2),
\] (3.33)
and
\[
\|\tilde{s}_0(\cdot) - \tilde{s}'(t, \cdot)\|_{L^p(D)}
\leq C\|\nu'(t, \cdot) - v_0(\cdot)\|_{L^p(D)} + C\|\nu'(t, \cdot) - v_0(\cdot)\|_{C^1(D_{1/2})}, \quad p \in [2, \infty).
\] (3.34)

Results of §2 guarantee that these quantities tend to 0 as \(\nu \searrow 0\), uniformly in \(t \in [0, T_0]\), under hypotheses weaker than (3.5). Results of Appendix A give
\[
\|1 - e^{\nu t}\|_{L^p(D)} \leq C(T_0)\nu^{1/2p},
\] (3.35)
for \(t \in [0, T_0]\). We have:

**Proposition 3.1** Assume that (3.5) holds. Also assume \(w_0 \in C^1(\overline{D})\), \(\beta \in BV_b(\mathbb{R})\), \(f^0, f' \in L^1_b(\mathbb{R})\) and \(\|f' - f^0\|_{L^1([0,T])} \to 0\), for each \(T < \infty\). Then, for each \(p \in [1, \infty)\),
\[
\|R_2(\nu, t, \cdot)\|_{L^p(D)} + \|R_3(\nu, t, \cdot)\|_{L^p(D)} \to 0 \quad \text{as} \quad \nu \searrow 0,
\] (3.36)
uniformly in \(t \in [0, T]\).

We turn our attention to \(R_1(\nu, t, x)\), given by (3.18), i.e., to the nature of the convergence
\[
e^{t(\nu\Delta - X)}W_0 \to e^{-tX}W_0,
\] (3.37)
as \(\nu \searrow 0\). Chapter 3 of [8] was devoted to such an analysis, in a mo are general setting, in which \(\overline{D}\) is replaced by a general compact Riemannian manifold with boundary \(\overline{\mathcal{O}}\), with Laplace-Beltrami operator \(\Delta\), and \(X\) is taken to be a smooth vector field on \(\overline{\mathcal{O}}\), tangent to the boundary \(\partial\mathcal{O}\) and satisfying \(\text{div} \, X = 0\). We recall some of these results.

Let us set
\[
U_\nu(t, x) = e^{t(\nu\Delta - X)}U_0(x).
\] (3.38)
The identity
\[
U^\nu(t) = e^{-tX}U_0 + \nu \int_0^t e^{-(t-s)X} \Delta U^\nu(s) \, ds
\] (3.39)
is useful once one has the following (Lemma 3.1.2 of [8]).
Lemma 3.2 There exists $K \in (0, \infty)$, independent of $\nu \in (0, 1]$, such that, if $U_0 \in \mathcal{D}(\Delta^2)$,
\begin{equation}
\|\Delta U_\nu(t)\|^2_{L^2} \leq e^{2Kt}\|\Delta U_0\|^2_{L^2}.
\end{equation}

This is proven by estimating
\begin{equation}
\frac{d}{dt}\|\Delta U_\nu(t)\|^2_{L^2} = 2 \Re (\Delta \partial_t U_\nu, \Delta U_\nu)_{L^2} = \cdots.
\end{equation}

The proof exploits the identity
\begin{equation}
\mathcal{D}((\nu \Delta - X)^2) = \mathcal{D}(\Delta^2).
\end{equation}

With this in hand, one proceeds to Proposition 3.1.3 of [8]:

Proposition 3.3 Given $p \in [1, \infty)$,
\begin{equation}
e^{t(\nu \Delta - X)}W_0 \rightarrow e^{-tX}W_0 \text{ as } \nu \rightarrow 0,
\end{equation}
in $L^p$-norm, for all $W_0 \in L^p(D)$.

Ingredients in the proof include the contraction property on $L^p(D)$ of $e^{t(\nu \Delta - X)}$, the validity of (3.43) on a dense subspace of $L^2(D)$, via Lemma 3.2, to get (3.43) for $p \in [1, 2]$, and then use of duality to get (3.43) for $p > 2$, first weak*, then, via uniform convexity, in norm.

Putting together Propositions 3.1 and 3.3, we have $L^p$ estimates on
\begin{equation}
w^\nu(t, x) - w^0(t, x) = W^\nu(t, x) - W^0(t, x)
= R_1(\nu, t, x) + R_2(\nu, t, x) + R_3(\nu, t, x).
\end{equation}

Proposition 3.4 Under the hypotheses of Proposition 3.1, as $\nu \searrow 0$,
\begin{equation}W^\nu(t, \cdot) \rightarrow W^0(t, \cdot), \text{ and hence}
\end{equation}
\begin{equation}w^\nu(t, \cdot) \rightarrow w^0(t, \cdot),
\end{equation}
in norm, in $L^p(D)$, for each $p \in [1, \infty)$.

We next discuss estimates in the spaces $\mathcal{V}^k(D)$, defined in (2.33)–(2.38). The following result (which extends Proposition 2.6) is Proposition 3.3.3 of [8]:

Proposition 3.5 For each $k \in \mathbb{Z}^+$, $\nu > 0$, $e^{t(\nu \Delta - X)}$ is a strongly continuous semigroup on $\mathcal{V}^k(D)$, and, with $B_k$ independent of $\nu \in (0, 1]$,
\begin{equation}\|e^{t(\nu \Delta - X)}W_0\|_{\mathcal{V}^k} \leq e^{tB_k}\|W_0\|_{\mathcal{V}^k}.
\end{equation}

The proof involves estimating a weighted sum of terms
\begin{equation}\frac{d}{dt}\|\mathcal{Y}^J U^\nu(t)\|^2_{L^2}, \quad |J| \leq k,
\end{equation}
and takes about 4 pages in [8]. From here, we get Proposition 3.3.4 of [8]:

Proposition 3.6 In the setting of Proposition 3.5,
\begin{equation}W_0 \in \mathcal{V}^k(D) \Rightarrow \lim_{\nu \searrow 0} e^{t(\nu \Delta - X)}W_0 \rightarrow e^{-tX}W_0,
\end{equation}
in norm, in $\mathcal{V}^k(D)$. 

16
We make some comments about the proof of this result. The boundedness result (3.47) plus the $L^2$-convergence from (3.43) imply convergence in (3.49), weak* in $V^k(D)$. To get norm convergence, one argues further. It suffices to get norm convergence on a dense subspace, e.g.,

$$C_0^\infty(D) \subset V^{2k}(D) \subset V^k(D).$$

(3.50)

Appendix A of [8] establishes the complex interpolation property

$$V^k(D) = [L^2(D), V^{2k}(D)]_{1/2}.$$  

(3.51)

Hence, for $f \in V^{2k}(D),$

$$\|e^{t(\nu\Delta - X)}f - e^{-tX}f\|_{V^k}$$

$$\leq \|e^{t(\nu\Delta - X)}f - e^{-tX}f\|_{L^2}^{1/2} \cdot \|e^{t(\nu\Delta - X)}f - e^{-tX}f\|_{V^{2k}}^{1/2}.$$  

(3.52)

The first factor on the right side of (3.52) tends to 0 as $\nu \searrow 0$, by Proposition 3.3, and the last factor is uniformly bounded as $\nu \searrow 0$ by (3.47), with $k$ replaced by $2k$.

We next discuss convergence of (3.45) in the spaces $V^k(D)$. For simplicity, we assume here that $g\nu, g^0 \equiv 0$,

$$W_\nu = \frac{\partial}{\partial t} (\nu\Delta - X_\nu) W_\nu, \quad W_\nu |_{\mathbb{R}^+ \times \partial D} = 0, \quad W_\nu(0, x) = W_0(x),$$

(3.54)

and $W^0$ by

$$\frac{\partial W^0}{\partial t} = -XW^0, \quad W^0(0, x) = W_0(x), \quad \text{i.e.,} \quad W^0(t) = e^{-tX}W_0.$$  

(3.55)

(Treating the general case simply involves one more use of Duhamel’s formula.) To set up the analysis, Chapter 4 of [8] defined a class $\tilde{X}^1$ of vector fields on $D$, depending on $t$ and $\nu$, as follows. Recall $X^1$, given by (2.35), and let $\{Y_j\}$ be a finite spanning set, as in (2.36). We say $Z_\nu \in \tilde{X}^1$ provided we can write

$$Z_\nu = \sum_j A^\nu_j(t, x)Y_j,$$

(3.56)

with

$$\{A^\nu_j(t, \cdot) : \nu \in (0, 1], \; t \in [0, T_0]\} \text{ bounded in } V^{\infty, \infty}(D),$$

(3.57)

for each $T_0 \in (0, \infty)$. In the current case of interest,

$$X_\nu = \tilde{s}_\nu(t, x) \frac{\partial}{\partial \theta},$$

(3.58)

Proposition 2.10 gives

$$X_\nu \in \tilde{X}_1.$$  

(3.59)

We also set

$$\tilde{X}^k = \text{Span}\{Z_\nu Y^I : Z_\nu \in \tilde{X}_1, \; Y^I \in \tilde{X}^{k-1}\}.$$  

(3.60)

The following results are established in Chapter 4 of [8]:

$$Z_\nu \in \tilde{X}_1, \; Y \in \tilde{X}_1 \implies [Z_\nu, Y] \in \tilde{X}_1,$$

$$P_\nu \in \tilde{X}^k, \; Y^I \in \tilde{X}^\ell \implies Y^I P_\nu \in \tilde{X}^{k+\ell},$$

(3.61)

and play a role in the demonstration of the next result (Proposition 4.1.5 of [8]).
Proposition 3.7 Assume $W_0 \in \mathcal{V}^k(D)$. Given that $X_\nu$ satisfies (3.59), there is a unique solution $W^\nu$ to (3.54), satisfying

$$W^\nu \in C([0, \infty), \mathcal{V}^k(D)) \cap C^\infty((0, \infty) \times \overline{D}),$$

and we have

$$\|W^\nu(t)\|_{\mathcal{V}^k} \leq e^{\nu B_k} \|W_0\|_{\mathcal{V}^k},$$

with $B_k$ independent of $\nu \in (0,1]$.

The proof involves estimating a weighted sum of terms

$$\frac{d}{dt} \|Y^J W^\nu(t)\|_{L^2}^2, \quad |J| \leq k,$$

and is a bit more elaborate than the proof of Proposition 3.5, bringing in the results of (3.61).

The uniform bounds (3.63) plus the $L^p$-norm convergence (3.45), with $p = 2$, imply the following, as shown in Propositions 4.2.1 and 4.2.4 of [8]:

Proposition 3.8 Retain the hypotheses of Proposition 3.1 (which imply (3.59)), and assume (3.53). Assume $W_0 \in \mathcal{V}^k(D)$. Then, as $\nu \searrow 0$,

$$W^\nu(t) \rightarrow e^{-tX}W_0,$$

in norm, in $\mathcal{V}^k(D)$.

As in Proposition 3.6, one first uses the uniform bounds mentioned above to get weak* convergence. Then the same argument used to go from weak* convergence to $\mathcal{V}^k$-norm convergence in Proposition 3.6, involving (3.50)–(3.52), works here.

We conclude this section with some complementary results. First there is the contraction property:

$$\|W^\nu(t, \cdot)\|_{L^p} \leq \|W_0\|_{L^p}, \quad 1 \leq p \leq \infty.$$  \hspace{1cm} (3.66)

Next, if also $W_0 \in \mathcal{V}^k(D)$ with $k > 1$, the result (3.65) implies

$$W^\nu(t, x) \rightarrow e^{-tX}W_0, \text{ locally uniformly in } D.$$  \hspace{1cm} (3.67)

In particular,

$$W_0 \in C^\infty(\overline{D}) \Rightarrow W^\nu(t) \rightarrow e^{-tX}W_0, \text{ boundedly and locally uniformly on } D.$$  \hspace{1cm} (3.68)

Combining (3.66) and (3.68) and using standard approximation arguments yields:

Proposition 3.9 In the setting of Proposition 3.8,

$$W_0 \in C(\overline{D}) \Rightarrow W^\nu(t) \rightarrow e^{-tX}W_0, \text{ boundedly and locally uniformly on } D.$$  \hspace{1cm} (3.69)
4 Further boundary layer estimates

In §3 we have seen various spaces in which

$$w'(t, x) - w^0(t, x) \to 0$$  \hspace{1cm} (4.1)$$

as $\nu \searrow 0$. Here we take a closer look at the boundary layers that form and prevent (4.1) from holding in sup norm. We work with the decomposition (3.21), i.e.,

$$w'(t, x) - w^0(t, x) = R_1(\nu, t, x) + R_2(\nu, t, x) + R_3(\nu, t, x),$$  \hspace{1cm} (4.2)$$

where, we recall,

$$R_1(\nu, t, x) = e^{t(\nu\Delta - X)}W_0 - e^{-tX}W_0, \hspace{1cm} (4.3)$$

$$R_2(\nu, t, x) = \int_0^t [f(4\nu(t-s)) - f^0(s)] ds \hspace{1cm} (4.4)$$

$$\quad + \int_0^t f(4\nu(t-s)) (e^{4\nu(t-s)} - 1) ds \hspace{1cm} (4.5)$$

$$\quad + \int_0^t (e^{4\nu(t-s)} - 1) d\beta(s), \hspace{1cm} (4.6)$$

$$R_3(\nu, t, x) = \int_0^t e^{4\nu(t-s)}(s_0 - s^\nu) \frac{\partial W(\nu)}{\partial \theta} ds. \hspace{1cm} (4.7)$$

We also recall that $W_0(x) = w_0(x) - \beta(0)$ and $W'(t, x)$ is given by (3.10)–(3.11). Note therefore that

$$\frac{\partial W'}{\partial \theta} = \frac{\partial w'}{\partial \theta}. \hspace{1cm} (4.8)$$

Of the three terms on the right side of (4.2), $R_2(\nu, t, x)$ is the easiest to analyze precisely. Proposition A.1 applied to $f \equiv 1$ gives

$$e^{4\nu(t-s)} - 1$$

$$\quad = - \sum_{j=0}^{2N} 2b_j(x)(4\nu(t-s))^{j/2} \frac{\varphi(s)}{\sqrt{4\nu(t-s)}} + \hat{R}_N(\nu(t-s), x), \hspace{1cm} (4.9)$$

where, for each $M, k \in \mathbb{N}$, there exists $N$ such that

$$\|\hat{R}_N(\nu(t-s), \cdot)\|_{C^k(\bar{D})} \leq C_{M,k}(\nu(t-s))^M, \quad 0 \leq s \leq t \leq T_0. \hspace{1cm} (4.10)$$

Thus, for example, the term (4.6) has the form

$$- \sum_{j=0}^{2N} 2b_j(x) \int_0^t (4\nu(t-s))^{j/2} E_j \left( \frac{\varphi(s)}{\sqrt{4\nu(t-s)}} \right) d\beta(s)$$

$$\quad + \int_0^t \hat{R}_N(\nu(t-s), x) d\beta(s). \hspace{1cm} (4.11)$$

We recall that

$$\varphi(x) = 1 - |x|, \quad b_0|_{\partial D} = 1, \hspace{1cm} (4.12)$$

19
and that $E_j$ is given by (A.39); in particular,

$$E_0(y) = \frac{1}{\sqrt{\pi}} \int_y^\infty e^{-s^2} ds. \tag{4.13}$$

Thus the principal term in (4.11) is

$$-2b_0(x) \int_0^t E_0 \left( \frac{1 - |x|}{\sqrt{4\nu(t - s)}} \right) d\beta(s), \tag{4.14}$$

and for $j \geq 1$ the $j$th term in (4.11) is

$$\leq C_j \nu^{j/2} t^{j/2}, \quad \text{uniformly in } x \in \overline{D}.$$  

The term (4.5) has a similar form as (4.6), and of course the term on the right side of (4.4) is completely elementary. We summarize.

**Proposition 4.1** We have

$$R_2(\nu, t, x) = -2b_0(x) \int_0^t E_0 \left( \frac{1 - |x|}{\sqrt{4\nu(t - s)}} \right) (d\beta(s) + f^\nu(s) ds) + \int_0^t [f^\nu(s) - f^0(s)] ds + O(\nu^{1/2} t^{1/2}), \tag{4.15}$$

uniformly in $x \in \overline{D}$, $t \in [0, T_0]$, $\nu \in (0, 1]$.

We turn to $R_1(\nu, t, x)$, given by (4.3). We assume $W_0 \in C^\infty(\overline{D})$. Results here are as precise as those for $R_2$, but somewhat more complicated. For the analysis, we use results from §§3.6–3.7 of [8]. The attack combines the use of layer potentials and semiclassical analysis. Before starting this attack, we first deal with the fact that the equation

$$\frac{\partial W_\nu}{\partial t} = \nu \Delta W_\nu - X W_\nu, \quad W_\nu|_{R^+ \times \partial D} = 0, \quad W_\nu(0, x) = W_0(x) \tag{4.16}$$

for

$$W_\nu(t) = e^{t(\nu \Delta - X)} W_0 \tag{4.17}$$

does not fit the pattern typically encountered in semiclassical analysis. One could regard (4.16) as semiclassical (with $\nu = \hbar^2$) if $X$ were zero order (which it is not) or if the vector field $X$ were accompanied by a factor $\hbar = \nu^{1/2}$. As it is, (4.16) is a more singular perturbation than that. The first step is to ameliorate this by considering

$$v_\nu(t, x) = e^{tX} e^{t(\nu \Delta - X)} W_0(x), \tag{4.18}$$

which solves

$$\frac{\partial v_\nu}{\partial t} = \nu L(t)v_\nu \quad \text{on } \mathbb{R}^+ \times D, \quad v_\nu|_{\mathbb{R}^+ \times \partial D} = 0, \quad v_\nu(0) = W_0 \tag{4.19}$$

with

$$L(t) = e^{tX} \Delta e^{-tX}, \tag{4.20}$$

a $t$-dependent family of second order, strongly elliptic operators with smooth coefficients on $\overline{D}$. (The functions $v_\nu$ and $u_\nu$, used below, are not to be confused with $v^\nu$ and $u^\nu$ from previous sections. Nor should $W_\nu$ be confused with $W^\nu$.) The solutions to (4.16) and (4.20) are related by the simple transformation

$$W_\nu(t) = e^{-tX} v_\nu(t). \tag{4.21}$$
We aim to express the solution to (4.19) as a sum of a “free space” solution and a layer-potential correction. To construct the free space solution, we put $D$ in a box in $\mathbb{R}^2$ and identify opposite edges, so $D$ is a domain with boundary in a compact manifold without boundary, say $M = \mathbb{T}^2 = \mathbb{R}^2/(4\mathbb{Z}^2)$. We extend $X$ from $D$ to a smooth vector field on $M$, and of course $\Delta$ extends to the Laplace operator on $M$, with its standard flat metric tensor. Then $L(t)$ in (4.20) is a well defined $t$-dependent family of second order strongly elliptic operators on $M$. Also extend $W_0 \in C^\infty(D)$ to $\tilde{W}_0 \in C^\infty(M)$. The free space solution is then $V_\nu(t)$, given by

$$
\frac{\partial V_\nu}{\partial t} = \nu L(t) V_\nu \text{ on } \mathbb{R}^+ \times M, \quad V_\nu(0, x) = \tilde{W}_0(x).
$$

(4.22)

The solution to (4.19) then has the form

$$
v_\nu(t, x) = V_\nu(t, x) - u_\nu(t, x), \quad t \geq 0, \ x \in D,
$$

where $u_\nu(t, x)$ satisfies

$$
\frac{\partial u_\nu}{\partial t} = \nu L(t) u_\nu \text{ on } \mathbb{R} \times D,
$$

$$
u_\nu = g_\nu = \chi_{\mathbb{R}^+}(t)V_\nu(t, x), \quad x \in \partial D,
$$

$$
u_\nu = 0 \text{ on } (-\infty, 0) \times D.
$$

(4.24)

The method of layer potentials is brought to bear to solve (4.24). This method involves the use of functions $H(\nu, s, t, x, y)$, defined as follows. First, the solution to (4.22) is given by

$$
V_\nu(t, x) = \int_M \tilde{W}_0(y)H(\nu, 0, t, x, y) \, dV(y),
$$

(4.25)

where $dV(y) = dy$ is the standard area element on $M = \mathbb{T}^2$. More generally, for $0 \leq s \leq t$,

$$
V_\nu(t, x) = \int_M V_\nu(s, y)H(\nu, s, t, x, y) \, dV_s(y),
$$

(4.26)

where $dV_s(y) = \sqrt{g(s, y)} \, dy$ is the pull-back of $dV$ via the flow generated by $X$, i.e., the Riemannian area element for $g_s$, the pull-back via this flow of the standard flat metric tensor on $M$.

It is not hard to analyze $V_\nu$ as a smooth function on $[0, \infty) \times M$, depending smoothly on $\nu \in [0, 1]$, given $\tilde{W}_0 \in C^\infty(M)$. Details can be found in §3.5 of [8]. Going from here to a layer potential analysis of $u_\nu$ in (4.24) requires an accurate approximation to the integral kernel $H(\nu, s, t, x, y)$. This was carried out in §3.6 of [8], in the more general context where $D$ is a smoothly bounded domain in a compact $n$-dimensional Riemannian manifold $M$, via techniques of semiclassical analysis. We summarize the results. We have

$$
H(\nu, s, t, x, y) = g(s, y)^{-1/2}K(\nu, s, t, x, x - y),
$$

(4.27)

where $K(\nu, s, t, x, x - y)$ has the form

$$
K(\nu, s, t, x, z) = \sum_{j=0}^N K_j(\nu, s, t, x, z) + R_N(\nu, s, t, x, z),
$$

(4.28)

where $R_N$ is increasingly negligible for large $N$ (cf. [8], Proposition 3.6.6), and the principal term $K_0(\nu, s, t, x, z)$ is given (with $n = 2$) by

$$
K_0(\nu, s, t, x, z) = \left(4\pi(1-s)^2 e^{-\frac{1}{2}g(s, t, x, x - y)} - \frac{dV}{g(s, t, x, x - y)} \right).
$$

(4.29)
Here $G(s,t,x)$ is a smooth positive-definite $n \times n$ (i.e., $2 \times 2$) matrix valued function of $(s,t,x) \in [0,\infty) \times [0,\infty) \times M$, whose construction involves a transport equation; cf. (3.6.79) of [8]. For $j \geq 1$, formulas for $K_j(\nu,s,t,x,z)$ are somewhat more elaborate variants of (4.29). They are given in (3.6.90) and (3.6.93) of [8]. The main point to take from these formulas is that, for $j \geq 1$, $K_j(\nu,s,t,x,z)$ is smaller and smoother than $K_0(\nu,s,t,x,z)$, which has a $\delta$-function type singularity in the limit $\nu \searrow 0$. In addition, these terms get progressively smaller and smoother as $j$ increases.

With these results in hand, we bring in the method of layer potentials to treat (4.24), following §3.7 of [8]. The double layer potential is given by

$$D_{\nu}h(t,x) = \nu \int_0^t \int_{\partial D} h(s,y) \frac{\partial H}{\partial n_{s,y}}(\nu, s, t, x, y) dS_s(y) ds.$$  (4.30)

Here $dS_s$ is the arc length on $\partial D$ induced by the metric tensor $g_s$, and $\partial/\partial n_{s,y}$ is the outward unit normal to $\partial D$ at $y \in \partial D$, determined by this metric tensor. The boundary trace relation for $D_{\nu}$ is

$$D_{\nu}h|_{\mathbb{R} \times \partial D} = \left( \frac{1}{2} I + \nu N_{\nu} \right) h,$$  (4.31)

for $\text{supp } h \subset \mathbb{R}^+ \times \partial D$, where

$$N_{\nu} h(t,x) = \int_0^t \int_{\partial D} h(s,y) \frac{\partial H}{\partial n_{s,y}}(\nu, s, t, x, y) dS_s(y) ds.$$  (4.32)

Thus the solution to (4.24) has the form

$$u_{\nu}(t,x) = D_{\nu} h_{\nu}(t,x),$$  (4.33)

provided $h_{\nu}$ solves

$$\left( \frac{1}{2} I + \nu N_{\nu} \right) h_{\nu} = g_{\nu},$$  (4.34)

with $g_{\nu}$ given in (4.24).

Solvability of (4.34), on any given $I = [0,T_0]$, for $\nu > 0$ small enough, is achieved as follows. From (4.27)–(4.29) and related results on $K_j$, one has

$$\|\nu N_{\nu} h\|_{L^\infty(I \times \partial D)} \leq C(I) \nu^{1/2} \|h\|_{L^\infty(I \times \partial D)}.$$  (4.35)

Cf. [8], (3.7.27). Hence, as long as $\nu^{1/2} \leq 1/2C(I)$, if $g_{\nu} \in L^\infty(I \times \partial D)$, the equation (4.34) is solved by

$$h_{\nu} = 2(I + 2\nu N_{\nu})^{-1} g_{\nu} = 2(I - 2\nu N_{\nu} + 4\nu^2 N_{\nu}^2 - \cdots) g_{\nu}.$$  (4.36)

We can take some finite sum of the series in (4.36) and have a rather small remainder. In particular,

$$\|h_{\nu} - 2g_{\nu}\|_{L^\infty(I \times \partial D)} \leq C(I) \nu^{1/2} \|g_{\nu}\|_{L^\infty(I \times \partial D)}.$$  (4.37)

Since, by (4.33),

$$u_{\nu} = 2D_{\nu} g_{\nu} + D_{\nu}(h_{\nu} - 2g_{\nu}),$$  (4.38)

it is useful to know that

$$\|D_{\nu} h\|_{L^\infty(I \times D)} \leq C\|h\|_{L^\infty(I \times \partial D)},$$  (4.39)
with \(C\) independent of \(\nu \in (0, 1]\); cf. (3.7.34) of [8]. Hence
\[
\|u_\nu - 2D_\nu g_\nu\|_{L^\infty(I \times D)} \leq C(I)\nu^{1/2}\|g_\nu\|_{L^\infty(I \times \partial D)} \\
\leq C'(I)\nu^{1/2}\|\tilde{W}_0\|_{L^\infty(M)};
\]
(4.40)
the latter inequality by (4.22)–(4.24) and the maximum principle.

The estimate (4.40) implies that \(2D_\nu g_\nu\) is a good enough approximation to \(u_\nu\), hence, via (4.23), that of \(v_\nu\), and hence, by (4.21), the boundary layer behavior of \(W_\nu = e^{t(\nu \Delta - X)}W_0\), given by (4.16)–(4.17). Thus we resolve the boundary layer behavior of \(R_1(\nu, t, x)\), at least to leading order. Taking more terms in the series in (4.36) leads to higher order approximation.

As for approximating \(u_\nu\) within \(O(\nu^{1/2})\) in sup norm, one can do this with a simplification of \(2D_\nu g_\nu\), namely \(2D_0\nu g_\nu\), where
\[
D_0h(t, x) = \nu \int_0^t \int \frac{\partial H_0}{\partial n_s}(\nu, s, t, x, y) dS_s(y) ds,
\]
(4.41)
where, in place of (4.27)–(4.28), we take
\[
H_0(\nu, s, t, x, y) = g(s, y)^{-1}K_0(\nu, s, t, x - y),
\]
(4.42)
again with \(K_0\) as in (4.29). We have, via estimates on \(K_j\) for \(j \geq 1\),
\[
\|D_\nu h - D_0^\nu h\|_{L^\infty(I \times D)} \leq C(I)\nu^{1/2}\|h\|_{L^\infty(I \times \partial D)}.
\]
(4.43)
Further estimates on \(V_\nu\) in (4.22)–(4.24) yield, for \(\delta > 0\),
\[
\|v_\nu - (W_0 - 2D_0^\nu W^b_0)\|_{L^\infty(I \times D)} \leq C(I)\nu^{1/2}\|W_0\|_{C^{1+\delta}(\overline{D})},
\]
(4.44)
where
\[
W^b_0 = \chi_{\mathbb{R}^+}(t)W_0|_{\partial D}.
\]
(4.45)
Cf. [8], Proposition 3.7.4. Recalling (4.16)–(4.18), we reach the following conclusion.

**Proposition 4.2** Assuming \(v_0, w_0 \in C^\infty(\overline{D})\),
\[
\|R_1(\nu, \cdot, \cdot) + 2e^{-tX}D_\nu^0 W^b_0\|_{L^\infty(I \times D)} \leq C(I)\nu^{1/2}\|W_0\|_{C^{1+\delta}(\overline{D})}.
\]
(4.46)

It remains to analyze \(R_3(\nu, t, x)\). Since \(W^\nu\) occurs on the right side of (4.7), we do not have as precise an analysis of \(R_3\) as we got for \(R_1\) and \(R_2\), but we are able to show the following.

**Proposition 4.3** In the setting of Proposition 4.2,
\[
R_3(\nu, t, x) \rightarrow 0,
\]
(4.47)
as long as
\[
\frac{1 - |x|}{\sqrt{\nu t}} \rightarrow \infty.
\]
(4.48)
We get this from the estimate (3.27), i.e.,

$$|R_3(\nu, t, x)| \leq \|\partial_\theta w_0\|_{L^\infty} \int_0^t e^{\nu(t-s)\Delta} |\tilde{s}_0 - \tilde{s}'(s)| \, ds,$$

(4.49)

which is then used to estimate the difference between $s_0(x) - s'(t, x)$.

$$|s_0(x) - s'(t, x)| \leq \psi\left(\frac{1 - |x|}{\sqrt{\nu t}}\right) + C\nu,$$

(4.50)

where $\psi(\lambda) \to 0$ as $\lambda \to \infty$. Such an estimate on $|v_0(x) - v'(t, x)|$

(4.51)

follows from (2.42)–(2.49), and then the estimate (4.50) follows by the arguments involving (2.60)–(2.71).

## A Heat semigroup boundary layers – a wave equation approach

Here we give some rather explicit formulas for the asymptotic behavior as $\nu \searrow 0$ of solutions to

$$\frac{\partial u'}{\partial t} = \nu \Delta u' \quad \text{on} \quad \mathbb{R}^+ \times \Omega,$$

(A.1)

satisfying

$$u'|_{\mathbb{R}^+ \times \partial \Omega} = 0, \quad u'(0, x) = f(x),$$

(A.2)

where $\Omega$ is a compact Riemannian manifold with smooth boundary and Laplace-Beltrami operator $\Delta$, and

$$f \in C^\infty(\overline{\Omega}).$$

(A.3)

Note that the solution to (A.1)–(A.2) is

$$u'(t, x) = e^{\nu t \Delta} f(x) = u(\nu t, x),$$

(A.4)

where $u = u'$ with $\nu = 1$. Thus the small $\nu$ analysis of (A.1)–(A.3) is just the small $t$ analysis of $e^{t\Delta} f$, for $f \in C^\infty(\overline{\Omega})$. Here $\Delta$ is the self-adjoint extension of the Laplace-Beltrami operator with domain $\mathcal{D}(\Delta) = H^2(\Omega) \cap H^1_0(\Omega)$.

We assume (without loss of generality) that $\Omega$ is a smoothly bounded open subset of $M$, a compact Riemannian manifold without boundary. Let $L$ denote the Laplace-Beltrami operator on $M$, and assume

$$f = \tilde{f}|_{\Omega} \quad \tilde{f} \in C^\infty(M).$$

(A.5)

Note that, for $x \in \Omega$, $t > 0$,

$$e^{t\Delta} f(x) = e^{tL} \tilde{f}(x) - U(t, x),$$

(A.6)

where $U(t, x)$ satisfies

$$\left(\partial_t - \Delta\right) U = 0 \quad \text{on} \quad \mathbb{R} \times \Omega,$n

$$U(t, x) = 0 \quad \text{for} \quad t < 0, \quad U(t, \cdot)|_{\partial \Omega} = \chi_{\mathbb{R}^+}(t)e^{tL} \tilde{f}|_{\partial \Omega}. $$

(A.7)

Standard hypoellipticity results give $U \in C^\infty(\mathbb{R} \times \Omega)$, hence, for $t \in (0, 1]$, $\overline{\mathcal{O}} \subset \subset \Omega$, $k, N \in \mathbb{N}$,

$$\|U(t, \cdot)\|_{C^k(\overline{\mathcal{O}})} \leq C_{N,k} t^N.$$

(A.8)
On the other hand, given $\tilde{f} \in C^\infty(M)$, the nature of the convergence of $e^{tL}\tilde{f}$ to $\tilde{f}$ is elementary and well known. From the fact that $e^{tL}\tilde{f} \in C^\infty([0, \infty) \times M)$ it follows that, for each $k, N \in \mathbb{N}$,

$$e^{tL}\tilde{f}(x) = \tilde{f}(x) + tL\tilde{f}(x) + \cdots + \frac{t^N}{N!} L^N \tilde{f}(x) + R_N(x),$$  \hspace{1cm} (A.9)

with

$$\|R_N\|_{C^k(M)} \leq C_{k,N}t^N, \quad 0 < t \leq 1.$$  \hspace{1cm} (A.10)

What remains is to analyze the precise nature of the boundary layer that forms for $U(t, x)$ as $t \searrow 0$, preventing the uniform convergence to 0 on $\overline{\Omega}$. One way to attack this problem is via the method of layer potentials. This was used in [6], and extended in [8] to study the more complicated problem in which (A.1) is replaced by $\partial u^\nu/\partial t = \nu \Delta u^\nu + Xu^\nu$, where $X$ is a smooth vector field on $\overline{\Omega}$, tangent to $\partial \Omega$. (These results are recalled in §4.) Here we bring in another method, based on a wave equation approach.

This approach starts with the identity

$$e^{t\Delta}f(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} \cos s\sqrt{-\Delta}f(x) \, ds,$$  \hspace{1cm} (A.11)

where $v(s, x) = \cos s\sqrt{-\Delta}f(x)$ solves the wave equation

$$(\partial_t^2 - \Delta)v = 0 \quad \text{on} \quad \mathbb{R} \times \Omega,$$

$v|_{\mathbb{R} \times \partial \Omega} = 0, \quad v(0, x) = f(x), \quad \partial_s v(0, x) = 0.$  \hspace{1cm} (A.12)

The identity (A.11) follows from the Fourier inversion formula and the spectral theorem (cf. [9], Chapter 8, and [2]). More generally than (A.11), one has

$$\varphi(\sqrt{-\Delta})f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(s) \cos s\sqrt{-\Delta}f \, ds,$$  \hspace{1cm} (A.13)

valid for even $\varphi \in \mathcal{S}(\mathbb{R})$, where $\hat{\varphi}(s) = (2\pi)^{-1/2} \int \varphi(\lambda)e^{-i\lambda s} \, d\lambda$. Taking $\varphi(\lambda) = e^{-t\lambda^2}$ yields (A.11).

Parallel to (A.6), we have, for $x \in \Omega, \ s \geq 0$,

$$\cos s\sqrt{-\Delta}f(x) = \cos s\sqrt{-L}\tilde{f}(x) - V(s, x),$$  \hspace{1cm} (A.14)

where $V(s, x)$ solves

$$(\partial_s^2 - \Delta)V = 0 \quad \text{on} \quad \mathbb{R} \times \Omega, \quad V(s, x) = 0 \quad \text{for} \quad s < 0,$$

$$V(s, \cdot)|_{\partial \Omega} = g(s, \cdot) = \chi_{\mathbb{R}^+}(s) \cos s\sqrt{-L}\tilde{f}|_{\partial \Omega}. \hspace{1cm} (A.15)$$

Also, parallel to (A.11),

$$e^{tL}\tilde{f}(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} \cos s\sqrt{-L}\tilde{f}(x) \, ds.$$  \hspace{1cm} (A.16)

Together (A.11), (A.14) and (A.16) and the evenness in $s$ of $e^{-s^2/4t}$ yield, for $t > 0, \ x \in \Omega,$

$$e^{t\Delta}f(x) = e^{tL}\tilde{f}(x) - \frac{2}{\sqrt{4\pi t}} \int_{0}^{\infty} e^{-s^2/4t} V(s, x) \, ds,$$  \hspace{1cm} (A.17)

hence, by (A.6),

$$U(t, x) = \frac{1}{\sqrt{\pi t}} \int_{0}^{\infty} e^{-s^2/4t} V(s, x) \, ds.$$  \hspace{1cm} (A.18)
We aim to analyze $V(s, x)$ and use this analysis in (A.18). The first step is to localize this analysis to small $s$. Given $a > 0$, pick an even function $\psi_1 \in C_0^\infty(\mathbb{R})$ such that $\psi_1(s) = 1$ for $|s| \leq a$, $0$ for $|s| \geq 2a$, and set $\psi_2(s) = 1 - \psi_1(s)$. We have

$$U(t, x) = U_1(t, x) + U_2(t, x),$$

$$U_j(t, x) = \frac{1}{\sqrt{\pi t}} \int_0^\infty \psi_j(s)e^{-s^2/4t}V(s, x)\, ds.$$  \hspace{1cm} (A.19)

In turn

$$e^{t\Delta} = \Phi_1^t(\sqrt{-\Delta}) + \Phi_2^t(\sqrt{-\Delta}),$$

where

$$\Phi_j^t(\sqrt{-\Delta}) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^\infty \psi_j(s)e^{-s^2/4t} \cos s\sqrt{-\Delta}\, ds.$$ \hspace{1cm} (A.21)

We see that for $t \in (0, 1]$, $k, N \in \mathbb{N}$,

$$\Phi_2^t(\lambda) \leq C_{k,N}t^N(1 + |\lambda|)^{-k},$$ \hspace{1cm} (A.22)

and hence

$$\|\Phi_2^t(\sqrt{-\Delta})f\|_{H^k(\Omega)} \leq C_{k,N}t^N\|f\|_{L^2(\Omega)}.$$ \hspace{1cm} (A.23)

Similarly

$$e^{t\mathcal{L}} = \Phi_1^t(\sqrt{-\mathcal{L}}) + \Phi_2^t(\sqrt{-\mathcal{L}}),$$ \hspace{1cm} (A.24)

with similar estimates, including

$$\|\Phi_2^t(\sqrt{-\mathcal{L}})\tilde{f}\|_{H^k(M)} \leq C_{k,N}t^N\|\tilde{f}\|_{L^2(M)}.$$ \hspace{1cm} (A.25)

Consequently we have

$$U_2(t, x) = \Phi_2^t(\sqrt{-\mathcal{L}})\tilde{f}(x) - \Phi_2^t(\sqrt{-\Delta})f(x),$$

$$\|U_2(t, \cdot)\|_{H^k(\Omega)} \leq C_{k,N}t^N\left(\|\tilde{f}\|_{L^2(\Omega)} + \|\tilde{f}\|_{L^2(M)}\right).$$ \hspace{1cm} (A.26)

Thus the boundary layer behavior of $U(t, x)$ is completely captured by $U_1(t, x)$. Hence we need a further analysis of $V(s, x)$ only for $s \in [0, 2a]$, where $a > 0$ can be taken as small as desired.

Note that in (A.15), $g$, defined on $\mathbb{R} \times \partial\Omega$, is supported in $\{s \geq 0\}$ and piecewise smooth, with a simple jump across $\{s = 0\}$. Finite propagation speed assures that for $s \geq 0$, $x \in \Omega$,

$$V(s, x) = 0 \quad \text{for} \quad \varphi(x) > s,$$ \hspace{1cm} (A.27)

where

$$\varphi(x) = \text{dist}(x, \partial\Omega).$$ \hspace{1cm} (A.28)

Let us pick $a > 0$ so small that

$$\overline{C} = \{x \in \overline{\Omega}: \varphi(x) \leq 2a\} \implies \varphi \in C^\infty(\overline{C}),$$ \hspace{1cm} (A.29)

and use this value of $a$ to pick $\psi_1$ and $\psi_2$ in (A.19). Then, for $s \in [0, 2a]$, $V(s, x)$ is given by a progressing wave expansion of the form

$$V(s, x) \sim \sum_{j \geq 0} a_j(s, x)(s - \varphi(x))^j_+,$$ \hspace{1cm} (A.30)
with coefficients $a_j \in C^\infty([0, 2a] \times \Omega)$, determined by certain transport equations. See [9], Chapter 6, §6. The meaning of (A.30) is that for each $N \in \mathbb{N}$,

$$V(s, x) = \sum_{j=0}^{N} a_j(s, x)(s - \varphi(x))^j_+ + R_N(s, x),$$  \hspace{1cm} (A.31)

where

$$R_N(s, x) = 0 \text{ for } \varphi(x) > s, \quad R_N \in C^N([0, 2a] \times \Omega).$$  \hspace{1cm} (A.32)

Writing

$$a_0(s, x) = a_0(\varphi(x), x) + \tilde{a}_1(s, x)(s - \varphi(x)),$$  \hspace{1cm} (A.33)

we can shift the latter term onto the $j = 1$ term in (A.31). Continuing this process, we have

$$V(s, x) = \sum_{j=0}^{N} b_j(x)(s - \varphi(x))^j_+ + R_N(s, x),$$  \hspace{1cm} (A.34)

(with slightly altered $R_N$, still satisfying (A.32)), valid on $[0, 2a] \times \Omega$, with $b_j \in C^\infty(\Omega)$. Inserting this into the formula for $U_1(t, x)$ given by (A.19), we have

$$U_1(t, x) = \sum_{j=0}^{N} \frac{b_1(x)}{\sqrt{\pi t}} \int_{0}^{\infty} e^{-s^2/4t} (s - \varphi(x))^j_+ \psi_1(s) ds$$
$$+ \int_{0}^{\infty} e^{-s^2/4t} R_N(s, x) \psi_1(s) ds.$$  \hspace{1cm} (A.35)

Recall the partition of unity $1 = \psi_1(s) + \psi_2(s)$, specified below (A.18). Elementary estimates show that

$$\int_{0}^{\infty} e^{-s^2/4t} (s - \varphi(x))^j_+ \psi_2(s) ds$$  \hspace{1cm} (A.36)

is rapidly decreasing as $t \searrow 0$, together with all $x$-derivatives, so the sum over $0 \leq j \leq N$ in (A.35) has the identical asymptotic behavior as $t \searrow 0$ as does

$$\sum_{j=0}^{N} b_j(x) W_j(t, x),$$  \hspace{1cm} (A.37)

$$W_j(t, x) = \frac{1}{\sqrt{\pi t}} \int_{0}^{\infty} e^{-s^2/4t} (s - \varphi(x))^j_+ ds.$$  \hspace{1cm} (A.38)

A change of variable gives

$$W_j(t, x) = 2(4t)^{j/2} E_j \left( \frac{\varphi(x)}{\sqrt{4t}} \right),$$  \hspace{1cm} (A.39)

where

$$E_j(y) = \frac{1}{\sqrt{\pi}} \int_{-y}^{\infty} e^{-s^2} (s - y)^j ds$$
$$= e^{-y^2} \int_{0}^{\infty} e^{-s^2 - 2ys} s^j ds.$$  \hspace{1cm} (A.39)

Using (A.32), one easily bounds the last integral in (A.35) by $CW_N(t, x)$. Consequently

$$U_1(t, x) = \sum_{j=0}^{N} 2b_j(x)(4t)^{j/2} E_j \left( \frac{\varphi(x)}{\sqrt{4t}} \right) + \tilde{R}_N(t, x),$$  \hspace{1cm} (A.40)
with
\[ \| \hat{R}_N(t, \cdot) \|_{C^0(\overline{\Omega})} \leq Ct^{N/2}. \] (A.41)

Similar arguments give estimates \[ \| \hat{R}_N(t, \cdot) \|_{C^k(\overline{\Omega})} \leq Ct^M, \] for each \( k, M \in \mathbb{N} \), if \( N \) is large enough.

Putting together (A.6), (A.9), (A.18), (A.19), (A.26), and (A.40), we obtain our main result:

**Proposition A.1** Given \( f \in C^\infty(\overline{\Omega}) \),
\[
e^t \Delta f(x) = f(x) + \sum_{k=1}^{N} \frac{t^k}{k!} \Delta^k f(x) - \sum_{j=0}^{2N} 2b_j(x)(4t)^{j/2} E_j \left( \frac{\varphi(x)}{\sqrt{4t}} \right) + \hat{R}_N(t, x),
\] (A.42)
where \( b_j \in C^\infty(\overline{\Omega}) \) are as in (A.34), and, for each \( M, k \in \mathbb{N} \), there exists \( N \) such that
\[
\| \hat{R}_N(t, \cdot) \|_{C^k(\overline{\Omega})} \leq C_{M, k} t^M, \quad t \in (0, 1].
\] (A.43)

**Remark.** It follows readily from (A.15) and (A.34) that \( b_j|_{\partial \Omega} = 0 \) when \( j \) is odd. Also \( b_0|_{\partial \Omega} = f|_{\partial \Omega} \), and \( E_0(0) = 1/2 \).

The following corollary, which follows by inspection of (A.42), is relevant for vorticity concentration.

**Corollary A.2** Given \( f \in C^\infty(\overline{\Omega}) \), we have
\[
\| \nabla e^t \Delta f \|_{L^1(\Omega)} \leq C_f, \quad \forall t \in (0, \infty).
\] (A.44)

**Remark.** Inspection of (A.42) also shows that such a uniform bound does not hold in any \( L^p \)-space with \( p > 1 \), unless \( f|_{\partial \Omega} = 0 \).

## B Poiseuille flow in a circular pipe

Given \( \alpha \in \mathbb{R} \setminus \{0\} \), the velocity field
\[
u(t, x, z) = \alpha(0, 1 - |x|^2)
\] (B.1)
is a well known example of a steady solution to the Navier-Stokes system
\[
\frac{\partial u^n}{\partial t} + \nabla u^n \cdot u^n + \nabla p^n = \nu \Delta u^n + F^n, \quad \text{div} u^n = 0,
\] (B.2)
on the infinite circular pipe
\[
\Omega = D \times \mathbb{R}, \quad D = \{ x \in \mathbb{R}^2 : |x| < 1 \},
\] (B.3)
an example of Poiseuille flow in a circular pipe (cf. [3], § 3.1). In such a case, $\partial_t u^\nu = 0$ and $\nabla u^\nu u^\nu = 0$. It is common to say that this flow is driven along the pipe by a uniform pressure gradient.

There are two ways to complete the description of how $u^\nu \equiv u_0$ solves (B.2). One is to set
\[
p^\nu(t, x, z) = -4\nu\alpha z, \quad F^\nu(t, x, z) = 0. \tag{B.4}
\]
The other is to set
\[
p^\nu(t, x, z) = 0, \quad F^\nu(t, x, z) = (0, 4\nu\alpha). \tag{B.5}
\]
Here we point out that this flow fits into the framework of our paper, in the setting of (B.5), but not in the setting of (B.4). Indeed, our analysis imposed the condition of periodicity in $z$ on all quantities, and hence passed to the quotient $D \times (\mathbb{R}/L\mathbb{Z})$, consequently obtaining solutions independent of $z$. However, $p^\nu(t, x, z) = -4\nu\alpha z$ is not periodic in $z$, and (B.4) is not well defined on $\Omega_L$, while (B.5) is well defined. Physically, it is appropriate to understand this flow, “driven by a uniform pressure gradient,” as driven by an external force. This favors the use of (B.5) over (B.4).

In fact, if we set $F^\nu \equiv 0$ in (B.2) and solve this, with initial data given by (B.1), as per the set-up in §1, we get, not a steady solution, but a solution $u^\nu(t, x, z)$ that decays to 0 as $t \nearrow \infty$. This is physically reasonable, since the energy dissipation due to $\nu \Delta u^\nu$ is not offset by energy input from an external force. We record what solution does arise.

The unique solution to (B.1)–(B.2) on $\mathbb{R}^+ \times \Omega_L$, with $F^\nu \equiv 0$, has the form
\[
u^\nu(t, x, z) = (0, w^\nu(t, x)), \tag{B.6}
\]
where $w^\nu$ solves (1.14) with $u^\nu \equiv 0$ and $f^\nu \equiv 0$, i.e.,
\[
\frac{\partial w^\nu}{\partial t} = \nu \Delta w^\nu, \quad \text{on } \mathbb{R}^+ \times D. \tag{B.7}
\]
The initial and boundary conditions are
\[
\nu^\nu(0, x) = w_0(x) = \alpha(1 - |x|^2), \quad \nu^\nu|_{\mathbb{R}^+ \times \partial D} = 0. \tag{B.8}
\]
In other words,
\[
\nu^\nu(t, x) = e^{\nu t} w_0(x). \tag{B.9}
\]

References


