1. (10 points) Prove that

\[ f \in L^p(\mathbb{R}^n) \iff \sum_{-\infty}^{+\infty} 2^{kp} \lambda_f(2^k) < \infty, \]

where \( \lambda_f \) is the distribution function of \( f \).

2. (10 points) Let \( f \in L^p(\mathbb{R}) \) and let \( u(x,y) = P_y * f(x), \ y > 0 \), be its harmonic extension to the upper half-plane \( \mathbb{R}^2_+ = \{ (x,y) \mid x \in \mathbb{R}, \ y > 0 \} \), where \( P_y \) is the Poisson kernel. Define the non-tangential maximal function \( u^*(x) \) by

\[ u^*(x) = \sup_{(x',y) \in \Gamma_x} |u(x',y)|, \]

where \( \Gamma_x \) is the unit cone with vertex at \( x \)

\[ \Gamma_x = \{ (x',y) \mid |x' - x| < y \}. \]

Using the material discussed in class, show that \( u^*(x) \leq CMf(x), \) a.e. \( x \in \mathbb{R}, \) where \( Mf \) being the Hardy-Littlewood maximal function and \( C \) a positive constant. Conclude that the non-tangential limit

\[ \lim_{(x',y) \to (x,0), (x',y) \in \Gamma_x} u(x',y) \]

exists and it is equal to \( f(x) \) for almost all \( x \in \mathbb{R}. \)

**Hint:** Use the results for approximate identities.

3. (10 points) Suppose \( f \) is a function supported on a ball \( B \subset \mathbb{R}^n \) of finite radius. Prove that \( Mf \in L^1(B) \) if \( |f| \log(2 + |f|) \in L^1(B). \)

**Hint:** Let \( Mf^1 = Mf \chi_{\{Mf(x) > 1\}} \) and write \( Mf = Mf^1 + (Mf - Mf^1) \). Use (without proof) that \( \forall \alpha > 0, \)

\[ |\{x \mid Mf(x) > \alpha\}| \leq \frac{2C}{\alpha} \int_{\{|f(x)| > \alpha/2\}} |f(x)| \, dx. \]