1. (10 points) Fix $0 < \alpha < 1$ and consider the series:

\[ \sum_{k=0}^{\infty} 2^{-\alpha k} e^{2\pi i 2^k x}. \]

This is a so-called *lacunary* Fourier series because all coefficients $a_n = 0$ if $n \neq 2^k$, $k = 0, 1, \ldots$. 

Prove that the series defines a (periodic) function $f$ on $[-1/2, 1/2]$ that is Hölder continuous of exponent $\alpha$, i.e.,

\[ \frac{|f(x + t) - f(x)|}{|t|^\alpha} \leq C_\alpha, \quad \forall -1/2 \leq x, t \leq 1/2, \quad t \neq 0, \]

for some constant $C_\alpha > 0$ (we then write $f \in C^\alpha$).

Conclude that, if a function $f \in C^\alpha$, then its Fourier coefficients $\hat{f}(n)$ decay at most like $\frac{1}{|n|^\alpha}$, as $|n| \to \infty$.

2. (10 points) Consider the function

\[ f(x) = \begin{cases} 
\pi - x, & 0 \leq x \leq \pi, \\
-\pi - x, & -\pi \leq x < 0,
\end{cases} \]

which is $2\pi$-periodic and of bounded variation on $[-\pi, \pi]$. We identify this interval with the one-dimensional torus. The Fourier series of $f$ is given by:

\[ 2 \sum_{n=1}^{+\infty} \frac{\sin nx}{n}. \]

This problem examines the *Gibbs phenomenon* for the function $f$.

(a) Restrict to the interval $0 \leq x \leq \pi$, and let $g_N(x) := S_N f(x) - f(x)$, where $S_N$ denotes the $N$th partial sum of the Fourier series. Show that $g'_N(x) = 2\pi D_N(x)$, where the Dirichlet kernel $D_N$ is given by

\[ D_N(x) := \frac{1}{\pi} \left( \frac{1}{2} + \cos(x) + \ldots + \cos(Nx) \right) = \frac{1}{2\pi} \frac{\sin((N + 1/2)x)}{\sin(x/2)}. \]
(This is the usual Dirichlet kernel with the change of variables $x = 2\pi t, -1/2 \leq t \leq 1/2$.)

(b) Let $\theta_N$ be the first critical point of $g_N$ for $x > 0$. Prove that $\theta_N = \pi/(N+1/2)$. Then show that

$$g_N(\theta_N) = \int_0^{\theta_N} \frac{\sin[(N + 1/2)x]}{\sin(x/2)} \, dx - \pi.$$

(c) Use part (b) to show that

$$\lim_{N \to \infty} g_N(\theta_N) = 2 \int_0^\pi \frac{\sin(x)}{x} \, dx - \pi \approx 0.562.$$

3. (10 points) Let $c_0 := \{(a_k), k \in \mathbb{Z} \mid |a_k| \to 0, |k| \to \infty\} \subset \ell^\infty$. This is closed subspace of $\ell^\infty$ and hence a Banach space with the Sup norm. Prove using the Open Mapping Theorem that the map $f \to \hat{f}$ cannot be surjective onto $c_0$. 

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