Math 312, Fall 2004
Practice Midterm 2
Total 40 pts+ 10 extra credit
“Prove” means give a careful, well-explained proof.
Put your name on the exam.
Good luck!
The amount of work for this practice midterm is expected to be more than 2 hours. The real midterm will be shorter.

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1a. Give the definition of supremum, infimum, maximum, minimum of $f(x)$ on an interval $I$. 
1b. Is the product of increasing functions an increasing function? Either prove it, or give a counterexample.
2. State the absolute convergence theorem for series and prove it.
3. Determine where the following series converge

a) \[ \sum \ln nx^n, \quad b) \sum \frac{1}{n^x}x^n, \quad c) \sum \frac{x^n}{\ln n}. \]

Use the n-th root test.

a) \[ \lim \left| a_n \right|^{1/n} = |x|. \]

If \( x = \pm 1 \), the series \( \sum (-1)^n \ln n \) diverges because \( a_n \not\to 0 \):

\[ \lim_{n \to \infty} \ln n = \infty. \]

b) \[ \lim \left| a_n \right|^{1/n} = 0, \quad R = \infty. \]

c) \[ \lim \left| a_n \right|^{1/n} = |x|, \quad R = 1. \]

If \( x = 1 \), the series \( \sum 1/\ln n \) diverges by the comparison test:

\[ \frac{1}{n} \leq \frac{1}{\ln n}. \]

If \( x = -1 \), the series \( \sum (-1)^n/\ln n \) converges by the Cauchy test.
4a. Suppose \( \{a_n\} \) and \( \{b_n\} \) are two sequences. Let

\[
s_n = \sum_{k=1}^{n} a_n
\]

be the n-th partial sum of \( \{a_n\} \). Prove the summation by parts formula: for \( m \leq n \)

\[
\sum_{k=m}^{n} a_k b_k = (s_n b_n - s_{m-1} b_m) + \sum_{k=m}^{n-1} s_k (b_k - b_{k+1}).
\]

\[
(s_n b_n - s_{m-1} b_m) + \sum_{k=m}^{n-1} s_k (b_k - b_{k+1})
\]

\[
= s_n b_n - s_{m-1} b_m + \sum_{k=m}^{n-1} s_k b_k - \sum_{k=m}^{n-1} s_k b_{k+1}
\]

\[
= s_n b_n + \sum_{k=m}^{n-1} s_k b_k - s_{m-1} b_m - \sum_{k=m+1}^{n} s_{k-1} b_k = \sum_{k=m}^{n} s_k b_k - \sum_{k=m}^{n} s_{k-1} b_k
\]

\[
= \sum_{k=m}^{n} (s_k - s_{k-1}) b_k = \sum_{k=m}^{n} a_k b_k.
\]
4b. Using the summation by parts formula prove that

\[ \sum_{n} \frac{\sin 2nx}{n} \]  

converges for all \( x \). Hint: you may use that for \( x \neq \pi m, m \in \mathbb{Z} \)

\[ \sum_{k=1}^{n} \sin 2nx = \frac{1}{2\sin x} (\cos x - \cos(2n+1)x). \]

When \( x = \pi m \) all \( \sin 2nx = 0 \). So the series converges to zero.
When \( x \neq \pi m \), we will show that (1) is Cauchy. We have

\[ b_k - b_{k+1} = \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}. \]

Using the summation by parts formula and the hint

\[ \sum_{k=m}^{n} \sin 2nx \frac{1}{n} = \frac{1}{2\sin x} \left( (\cos x - \cos(2n+1)x) \frac{1}{n} \right) \]

\[ - (\cos x - \cos(2m-1)x) \frac{1}{m} + \sum_{k=m}^{n-1} (\cos x - \cos(2k+1)x) \frac{1}{k(k+1)} \).

Since \( |\cos \alpha| \leq 1 \) and \( 1/k(k+1) \leq 1/k^2 \), by triangle inequality

\[ \left| \sum_{k=m}^{n} \sin 2nx \frac{1}{n} \right| = \frac{1}{|\sin x|} \left( \frac{1}{n} + \frac{1}{m} + \sum_{k=m}^{n-1} \frac{1}{k^2} \right) . \]

Since

\[ \sum \frac{1}{k^2} \]

converges and \( 1/n \to 0 \), as \( n \to \infty \), given \( \varepsilon > 0 \), there exists \( N \), such that for any \( m, n \geq N \)

\[ \frac{1}{n} < \varepsilon \frac{|\sin x|}{3}, \frac{1}{m} < \varepsilon \frac{|\sin x|}{3}, \sum_{k=m}^{n-1} \frac{1}{k^2} < \varepsilon \frac{|\sin x|}{3}. \]

Altogether

\[ \left| \sum_{k=m}^{n} \frac{\sin 2nx}{n} \right| < \varepsilon. \]
5 (Extra credit). Prove
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \Rightarrow \lim_{n \to \infty} |a_n|^{1/n} = L.
\]

Deduce that the n-th root test is stronger than the ratio test, in the sense that if a series satisfies the hypothesis of the ratio test, it will also satisfy the hypothesis of the n-th root test. Give an example of the series, for which the ratio test is inconclusive, whereas the n-th root test determines the convergence property of the series.

Suppose \( \lim_{n \to \infty} |a_{n+1}|/|a_n| = L \). It implies that for a given \( \varepsilon > 0 \), there exists \( N \) such that \( |a_n|/(L - \varepsilon) < |a_{n+1}| < |a_n|(L + \varepsilon) \), for all \( n > N \). By induction we can prove that there exist two positive constants \( c_1 \) and \( c_2 \) so that for any \( n > N \):
\[
|a_n| < c_1(L - \varepsilon)^n < c_2(L + \varepsilon)^n.
\]
Since for any \( c_1 \) and \( c_2 \)
\[
\lim_{n \to \infty} (c_1(L - \varepsilon)^n)^{1/n} = (L - \varepsilon) \lim_{n \to \infty} c_1^{1/n} = L - \varepsilon, \quad \lim_{n \to \infty} (c_2(L + \varepsilon)^n)^{1/n} = L + \varepsilon,
\]
there exists \( N_1 \) so that for all \( n > N_1 \)
\[
(c_1(L - \varepsilon)^n)^{1/n} > L - 2\varepsilon, \quad (c_2(L + \varepsilon)^n)^{1/n} < L + 2\varepsilon.
\]
Hence for all \( n > N_1 \)
\[
L - 2\varepsilon < |a_n|^{1/n} < L + 2\varepsilon.
\]
Therefore by the \( K - \varepsilon \) principle \( \lim |a_n|^{1/n} = L \).

Let us now construct a counterexample. Take
\[
a_n = x_n, \quad \text{where} \quad x_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}.
\]

We know that (p.52 Example 4.2) \( \sum_{n=1}^{\infty} (-1)^{n+1}/n = \ln 2 < 1 \). Hence \( \lim_{n \to \infty} |a_n|^{1/n} = \lim_{n \to \infty} x_n = \ln 2 \). This guarantees that the series converges. The ratio test is, however, inconclusive:
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = x_{n+1} \left( 1 + \frac{(-1)^{n+2}}{(n+1)x_n} \right)^n.
\]
On the one hand, when \( n = 2m \) is even
\[
\lim_{m \to \infty} \left| \frac{a_{2m+1}}{a_{2m}} \right| = \lim_{m \to \infty} x_{2m+1} \left( 1 + \frac{1}{(2m+1)x_n} \right)^{2m} = \ln 2 \lim_{m \to \infty} \left( 1 + \frac{1}{(2m+1)x_n} \right)^{2m/(2m+1)x_n} = \ln 2 \exp(1/\ln 2).
\]
On the other hand, when \( n = 2m - 1 \) is odd
\[
\lim_{m \to \infty} \left| \frac{a_{2m}}{a_{2m-1}} \right| = \lim_{m \to \infty} x_{2m} \left( 1 - \frac{1}{2mx_{2m-1}} \right)^{2m-1} = \ln 2 \exp(-1/\ln 2).
Some further practice questions.
Suppose \( \sum a_n \) converges \( a_n \neq 0 \). Decide whether the following statement is true or false and then prove that you are correct.

a) \( \sum (-1)^n a_n \) converges
b) \( \sum a_n x^n \) converges for \( |x| < 1 \).
c) \( \sum a_n^3 \) converges.
d) \( \sum 1/a_n \) diverges.

a) False \( a_n = (-1)^n/n \).
b) True. Since \( \sum a_n \) converges \( |a_n| \leq K \). Hence \( |a_n||x^n| \leq K|x^n| \).

Since the geometric series \( \sum |x|^n \) converges for \( |x| < 1 \), by comparison test \( \sum a_n x^n \) converges for \( |x| < 1 \).

c) False 
\( \sum a_n = 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{21/3} \left( 1 - \frac{1}{2} - \frac{1}{2} \right) + \frac{1}{31/3} \left( 1 - \frac{1}{2} - \frac{1}{2} \right) + \cdots = 0. \)

\( \sum a_n^3 = 1 - \frac{1}{2^3} - \frac{1}{2} \left( 1 - \frac{1}{2^3} - \frac{1}{2} \right) + \frac{1}{3} \left( 1 - \frac{1}{2^3} - \frac{1}{2} \right) + \cdots = \frac{3}{4} \sum \frac{1}{n} = \infty. \)

d) True. \( a_n \not\to 0. \)

Using power series representation for the geometric series
\[ \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \]
find power series representation for \( \ln(1+x), 0 < x < 1 \). Prove it rigorously.

\[ \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}. \]

Let 
\[ S_k = \sum_{n=0}^{k} \frac{(-1)^n}{n+1} x^{n+1}, s_k = \sum_{n=0}^{k} (-1)^n x^n \]

Then
\[ |\ln(1+x) - S_k| = \left| \int_0^x \frac{dt}{1+t} + \int_0^k s_k(t) dt \right| \leq \int_0^x \left| \sum_{n=0}^{\infty} (-1)^n t^n - \sum_{n=0}^{k} (-1)^n t^n \right| dt \]
\[ = \int_0^x \left| \sum_{n=k+1}^{\infty} (-1)^n t^n \right| dt = \int_0^x t^{n+1} \left| \sum_{n=0}^{\infty} (-1)^n t^n \right| dt = \int_0^x t^{n+1} \frac{dt}{1+t} \leq \int_0^x \frac{t^{n+1}}{n+2} dt = \int_0^x \frac{t^{n+1}}{n+2} dt \leq \frac{x^{n+2}}{n+2}. \]

When \( 0 < x < 1 \)
\[ |\ln(1+x) - S_k| < \frac{x^{n+2}}{n+2} \to 0, \]
as \( n \to \infty. \)