Math 312, Fall 2004
Practice midterm 1
Total 40 pts + 10 extra credit
Please, show the details of your work.
Put your name on the exam.
Good luck!
1a. Give the definition of an increasing sequence.
1b. Formulate the negative statement without using "not":
Sequence \( \{a_n\} \) converges to zero.

**Solution** There is a positive number \( C \), and a subsequence \( \{a_{n_i}\} \) so that

\[ |a_{n_i}| \geq C. \]
2. State the uniqueness theorem for limits and prove it.
3. Evaluate the limits and prove you are correct:

\[ a_{n+1} = a_n^2, \quad a_1 = -1/2, \ n \in \mathbb{N}, \]
\[ b_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^n \frac{1}{n+1}, \ n \in \mathbb{N}, \]
\[ c_n = -\frac{n}{n^2 + 1}, \ n \in \mathbb{N}. \]

**Solution** a) \( a_n \to 0. \) Let us show by induction that
\[ a_n = \frac{1}{2^k - 1}, \ n \geq 2. \]

Basis: \( n = 2: \)
\[ a_2 = \frac{1}{2^2} = \frac{1}{4}. \]

Inductive step: by induction
\[ a_{k+1} = a_k^2 = \left( \frac{1}{2^k - 1} \right)^2 = \frac{1}{2^{2k} - 2} = \frac{1}{2^{2k}}. \]

Given \( \varepsilon > 0 \)
\[ |a_n - 0| < \varepsilon, \]
when
\[ \frac{1}{2^{2n}} < \varepsilon \iff 2^n \ln 2 > -\ln \varepsilon \iff n > \ln \left( \frac{\ln \varepsilon}{\ln 2} \right) / \ln 2. \]

For b) see book. Geometric series.

C) \( c_n \to 0. \) Let us estimate the error
\[ \left| -\frac{n}{n^2 + 1} - 0 \right| \leq \frac{1}{n} \]

Hence
\[ \left| -\frac{n}{n^2 + 1} - 0 \right| \leq \varepsilon, \ \text{when} \ n \geq \frac{1}{\varepsilon}. \]
4. Suppose \( a_n \to 0 \) as \( n \to \infty \), \( b_n \) is bounded. Prove that for any sequence \( \{c_n\} \)

\[
\frac{a_n b_n}{1 + c_n^2} \to 0,
\]
as \( n \to \infty \).

**Solution**

\[
\left| \frac{a_n b_n}{1 + c_n^2} - 0 \right| \leq |a_n| \frac{|b_n|}{1 + c_n^2}
\]

Since \( b_n \) is bounded, there exists a constant \( B \geq 0 \), so that \( |b_n| \leq B \) for all \( n \). For any \( c_n \) we also have

\[
1 + c_n^2 \geq 1 \iff \frac{1}{1 + c_n^2} \leq 1.
\]

Hence

\[
|a_n| \frac{|b_n|}{1 + c_n^2} \leq B |a_n|
\]

Since \( a_n \to 0 \). Given \( \varepsilon > 0 \), \( |a_n| < \varepsilon \), for \( n \geq N \). Hence

\[
|a_n| \frac{|b_n|}{1 + c_n^2} < B \varepsilon,
\]

for \( n \geq N \). By the \( K - \varepsilon \) principle,

\[
\frac{a_n b_n}{1 + c_n^2} \to 0,
\]
as \( n \to \infty \).
5 (Extra credit). If \( a_n \to L \), and \( b_n \) lies between \( a_n \) and \( a_{n+1} \), prove \( b_n \to L \).

Remark: Note that "between" does not tell you in which direction the inequalities go.

**Solution** Define two new sequences

\[
m_n = \min(a_n, a_{n+1}), \quad M_n = \max(a_n, a_{n+1}).
\]

We have

\[
m_n \leq b_n \leq M_n.
\]

Hence, by squeeze theorem, we just need to show that

\[
m_n \to L, \quad M_n \to L,
\]

as \( n \to \infty \). The latter follows from the definition of the limit. We know that given \( \varepsilon > 0 \) \( |a_n - L| < \varepsilon \) when \( n > N \). Therefore, using the same \( N \) we also have

\[
|m_n - L| \leq \max(|a_n - L|, |a_{n+1} - L|) < \varepsilon,
\]

and

\[
|M_n - L| \leq \max(|a_n - L|, |a_{n+1} - L|) < \varepsilon.
\]