On the dynamics of the Gross-Pitaevskii equation

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IMPA, Rio de Janeiro, School Around vortices: from continuum to quantum Mechanics
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Based on various works with: Raphaël Danchin (Créteil), Philippe Gravejat (Ecole Polytechnique, CMLS) Jean-Claude Saut (Orsay) and Didier Smets (Paris 6)
The Gross-Pitaevskii equation

The \((GP)\) equation is a nonlinear Schrödinger equation of defocusing type. It writes in its simplest form as

\[
(GP) \quad i \frac{\partial \psi}{\partial t} + \Delta \psi = \psi(|\psi|^2 - 1) \quad \text{on} \quad \mathbb{R}^N
\]

The emphasis is put on the "non standard" boundary conditions at infinity

\[|\psi|^2 \to 1 \quad \text{as} \quad |x| \to +\infty.\]

The \((GP)\) equation is a model in various areas of

- low temperature physics: Bose-condensation, Superfluids, etc....\(\psi\) represents a condensed wave-function, \(|\psi|^2\) its local density of matter
- nonlinear optics.
More realistic models are provided by the more general model

\[ (WGP) \quad i \frac{\partial \Psi}{\partial t} + \Delta \Psi = \Psi [W \ast (|\Psi|^2 - 1)] \]

where $W$ is a function which models the interactions between particles.

$(GP)$ corresponds to the case $W = \delta_0$ and models hence only local interactions, whereas more general $W$ allows also for long-range interactions and rotons.

The $(GP)$ equation is modeled on the Schrödinger equation and hence is dispersive.
An important preliminary remark

The form of \((GP)\) is very close to the cubic defocusing NLS equation

\[
(NLS) \quad i \frac{\partial \xi}{\partial t} + \Delta \xi = \xi |\xi|^2
\]

where one takes usually as boundary conditions at infinity

\[|\xi|^2 \to 0 \quad \text{as} \quad |x| \to +\infty.\]

The properties of the solutions are however very different. Whereas the defocusing \((NLS)\) flow is essentially governed by scattering governed by the linear Schrödinger equation, and exhibits no special solutions, like

- solitons,
- standing waves,
- etc...

we will see that \((GP)\) possesses a very rich dynamics.
Remark. If $\xi$ is a solution to the focusing (NLS) equation above then

$$\Psi = e^{ix_1} \xi$$

is a solution to (GP) with the same type of boundary conditions at infinity.

We will next present several mathematical results which partially describe the rich dynamics of the (GP) equation. A striking fact is that it connects to numerous classical evolution PDE:

- The compressible and incompressible Euler equation of fluid dynamics
- The linear wave equation
- The KDV (Korteweg de Vries) or KP (Kadomtchev Petaschvili) equations

This is for a large part related to the hydrodynamical formulation of the equation.
Hydrodynamical formulation

We discuss here the form of the equation when the wave function does not vanish, that is assuming

$$\psi(x, t) \neq 0, \ \forall x \in \mathbb{R}^N, \ t \in [0, T].$$

we may then write (Madelung transformation)

$$\psi = \sqrt{\varrho} \exp(i \varphi).$$

where the phase $\varphi$ is continuous. Setting $v = 2 \nabla \varphi$, we obtain

$$\begin{cases}
\partial_t \varrho + \text{div}(\varrho v) = 0, \\
\varrho(\partial_t v + v \cdot \nabla v) + \nabla \varrho^2 = 2 \varrho \nabla \left( \frac{\Delta \varrho}{\rho} \right).
\end{cases} \quad (\text{HGP})$$

termed the hydrodynamical form of (GP).
Indeed, neglecting the term $\nabla \tilde{p} \equiv \nabla \left( \frac{\Delta \rho}{\rho} \right)$ one recovers the \textbf{compressible Euler} equation

\[
\begin{aligned}
\begin{cases}
\partial_t \varrho + \text{div}(\varrho \mathbf{v}) = 0, \\
\varrho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla \varrho^2 = 0.
\end{cases}
\end{aligned}
\tag{Euler}
\]

with pressure law

\[p(\varrho) = \varrho^2.\]

The previous discussion is of course \textit{meaningful} as long as $\psi$ \textbf{does not} vanish.

\textbf{Remark 1:} It turns out that the zeroes of $\Psi$ termed sometimes the \textit{vorticity} play a role similar to \textit{singularities focusing} equations. The energy estimates yields no information on the phase.
Remark 2: The hydrodynamical formulation is relevant also for the standard defocusing cubic NLS. However, it is less accurate if the solutions goes to zero at infinity, then there is little hope to control the points where $\xi$ vanishes.
At least formally ($GP$) conserves various quantities:

- the Hamiltonian, often termed the Ginzburg-Landau energy

$$E(\Psi) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \Psi|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (1 - |\Psi|^2)^2 \equiv \int_{\mathbb{R}^N} e(\Psi).$$

We are mostly interested in finite energy solutions

- the momentum

$$\vec{P}(\Psi) = \frac{1}{2} \int_{\mathbb{R}^N} \langle i \nabla \Psi, \Psi - 1 \rangle$$

- the mass

$$m(\Psi) = \frac{1}{2} \int_{\mathbb{R}^N} (|\Psi|^2 - 1).$$

Similar conservation laws are present for ($WGP$).
Remark If we use the Madelung transformation

\[ \psi = \sqrt{\varrho} \exp i \varphi \]

then the momentum has the simple formal expression

\[ \vec{P}(\Psi) = \frac{1}{2} \int_{\mathbb{R}^N} \varrho \left( \nabla \varphi \right). \]

whereas the mass has the formal expression

\[ m(\psi) = \frac{1}{2} \int_{\mathbb{R}^N} \left( \varrho - 1 \right) . \]
Finite energy solutions

We are mainly interested in finite energy solutions that is for which

\[ E(\psi(\cdot, t)) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi(\cdot, t)|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (1 - |\psi(\cdot, t)|^2)^2 < \infty. \]

Here, it is the finitness of the integral of the potential

\[ \int_{\mathbb{R}^N} V_{GL}(\psi(\cdot, t)) \equiv \int_{\mathbb{R}^N} ((1 - |\psi(\cdot, t)|^2)^2 \]

which induced the boundary condition at infinity

\[ |\psi(\cdot, t)| \to 1 \quad \text{as} |x| \to \infty \]
Notice that the Ginzburg-Landau potential achieves its minimum on the circle

$$S^1 = \{ s \in \mathbb{C}, |z| = 1 \} = \{ e^{i\theta}, \theta \in \mathbb{R} \},$$

also termed the vacuum set.

Hence there is one degree of freedom at the minimum, the phase $$\theta.$$
Special solutions

They play an important role in the dynamics:

- **Vacuum states**, i.e. minimizers of the **potential**, of the form \( \exp i\theta \), \( \theta \in [0, 2\pi] \)
- **vortices** which are other stationary solutions of **infinite energy**
- **traveling waves**.
Vortices are related to the points where the function $\Psi$ vanishes

$$\Psi(x, t) = 0,$$

In dimension two there are radial solutions which are stationary for $(GP)$

$$\psi_d(x, t) = f(|x|) \exp id\theta, \quad d \in \mathbb{Z},$$

The GL-energy diverges (at infinity), but combinations which total winding number equal to zero

$$\psi = \prod_{i=1}^{d} \psi_{d_i}(x - a_i) \quad \text{with} \quad \sum_{i=1}^{d} d_i = 0,$$

have finite energy. For such configurations the motion of the points $a_i(t)$ is governed by the point-vortex equation, which appears for instance in fluid dynamics for the incompressible Euler equation (Colliander-Jerrard (1999)).
Graph of a degree one stationary vortex
Vortex-antivortex pair has finite energy
Solitary waves are special solutions of the form

\[ \Psi(x, t) = U_c(x - \vec{c}t), \]

where \( \vec{c} \in \mathbb{R}^N \) and \( U_c : \mathbb{R}^N \rightarrow \mathbb{C} \) is a non-trivial function. In view of invariance by rotation one may choose \( \vec{c} = ce_1, \ c \geq 0 \).

In dimension \( N = 1 \) solutions can be integrated explicitly to yield the branch, with \( 0 \leq c < \sqrt{2} \)

\[ U_c(x) \equiv \sqrt{\frac{2 - c^2}{2}} \ \text{th} \left( \frac{\sqrt{2 - c^2}}{2} x \right) - i \frac{c}{\sqrt{2}}, \]
We may draw the graph

\[ E = \mathcal{E}(p) \]

Each point of the curve represents a non-constant solution \( U_c \) of energy \( E(U_c) \) and scalar momentum \( p(U_c) \). The speed of the solution \( U_c \) is given by the slope of the curve. The curve \( E = \mathcal{E}(p) \) is increasing and strictly concave, and lies below the line \( E = \sqrt{2}p \).

The solution actually minimize the energy for fixed momentum

- \( \implies \) orbital stability \([\text{Lin, B- Gravejat-Saut, Gerard-Wang}]\)
- No scattering theory possible
First relation with KdV

The waves $U_c$ are related to the solitons of the Korteweg-de Vries equation. Set $\varepsilon = \sqrt{2 - c^2}$, $\eta_c \equiv 1 - |U_c|^2$ and consider the scaled function

$$N_\varepsilon(x) = \frac{1}{\varepsilon^2} \eta_c \left( \frac{x}{\varepsilon} \right), \text{ and } U_c(x) = 1 + \varepsilon^2 w_\varepsilon(\varepsilon x)$$

We have $N_\varepsilon(x) = N(x) \equiv \frac{1}{2\cosh^2 \left( \frac{x}{2} \right)}$ and $U_c(x) = 1 + \varepsilon^2 w_\varepsilon(\varepsilon x)$, where $w_\varepsilon$ is bounded. A remarkable property of $N$ is that it represents a classical soliton to the Korteweg-de-Vries equation

$$\partial_t w + \partial_x^3 w + 6w\partial_x w = 0. \quad \text{(KdV)}$$
the kink solution $c = 0$
Solitary waves continued

In higher dimensions solitary waves may be obtained through variational methods, e.g. constraint minimization. In dimension $N = 2$, the function $E_{\text{min}}$ has the following graph:

- Each point of the graph corresponds to a solitary wave ([B-Gravejat-Saut, 2009])
- Orbital stability [Chiron-Maris, 2011]
- Vortices appear at high energy (B-Saut, 1999)
- Low energy waves are related to KP-Solitons modulo a scaling and stretching of coordinates.
- No scattering possible
High energy solitary waves $N = 2$

The speed $c \to 0$ as energy increases and

$$E(v) \sim 2\pi \ln(c), \text{ and } p(v) \sim \frac{2\pi}{c}, \text{ as } c \to 0.$$
Low energy solitary waves $N = 2$

Here we consider the other end of the curve, namely solutions with small energy and momentum.

Since $c$ is the slope of the curve, it follows that

$$c \to \sqrt{2}$$
we introduce the small parameter $\varepsilon = \sqrt{2 - c^2}$ and expand $
abla_c = 1 - |U_c|^2$ as

$$\nabla_c(x) = 1 + \varepsilon^2 N\varepsilon(\varepsilon x_1, \varepsilon^2 x_2).$$

**Proposition ([B-Saut-Gravejat 2009])**

$N\varepsilon$ converges to a ground state $N_0$ of the $(KP I)$ equation

$$\partial_t u + u\partial_1 u + \partial_3^3 u - \partial_1^{-1}(\partial_2^2 u) = 0.$$  

- KdV like terms
- tranverse perturbation
A special known soliton solution for \((KP)\) : the lump-soliton for \((KP)\)

\[
\psi_{\ell}(x_1, x_2) = 24 \frac{3 - x_1^2 + x_2^2}{(3 + x_1^2 + x_2^2)^2}.
\]

It is however not known if this solution is a ground state: recall that the \((KP-I)\) equation is hamiltonian

\[
E_{KP}(u) = \frac{1}{2} \int_{\mathbb{R}^2} (\partial_1 u)^2 + (\partial_1^{-1}(\partial_2 u))^2 - \frac{1}{6} \int_{\mathbb{R}^2} u^3
\]

and that the \(L^2\) norm is conserved.
solitary waves for N=3

The figure is quite different: minimizing solitary waves exists up to a momentum $p_0$, but solutions exist for any speed $0 < c < \sqrt{2}$ [Maris 2009], see upper branch. No solutions for $c \geq \sqrt{2}$ (Gravejat 2004). Allows for

- Scattering for low energy (See [Gustafson-Nikanishi-Tsai (2008)])
The Cauchy problem

Global well-posedness in the affine space $1 + H^1(\mathbb{R}^N)$, for $N = 1, 2, 3$ is obtained by classical tools (integral Duhamel formulation + dispersive Strichartz + bound due to energy conservation) (see B-Saut, 99).

Writing $\psi = 1 + u$, one obtains indeed the equation for $u$

$$iu_t + \Delta u - 2\Re u = u^2 + 2|u|^2 + |u|^2 u.$$ 

A good global existence theory in the (not affine) energy

$$\mathcal{X}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R}^2 \text{ s.t } E(u) < +\infty \} \supset 1 + H^1(\mathbb{R}^N)$$ 

is provided in the works of

- P. Gerard and C. Gallo, for $1 \leq N \leq 3$, for $(GP)$
- for $N = 4$ by Killip, Oh, Pocovnicu and Visan for $(GP)$
- A. De Laire for $(WGP)$ for $1 \leq N \leq 3$.

Solutions conserve the Ginzburg-Landau energy.
Main questions

**Question 1**
Perturbing slightly a traveling wave, does one stay close, as time tends to $+\infty$, to a translated traveling wave?

Two kind of notions:
- Orbital stability
- asymptotic stability

**Question 2**
An important goals is understand how vorticity is created by the equations and how it evolves and interacts with waves provided by the hydrodynamical formulation.
Stability issues

Notice that in our framework, we have families (actually curves) of special solutions.

**Orbital stability means:** Perturbing slightly a special solution in a given family (traveling wave, standing wave, etc...) does the solution stay close (up to translation) to a nearby special solution.
Asymptotic stability means: Perturbing slightly a special solution in a given family (traveling wave, standing wave, etc...) does the solution converge to a nearby special solution.

Convergence has to be understand in the sense of some local norms due to Reversibility, conservations of energy, momentum, ...
Orbital Stability in 1D

To illustrate orbital stability, we have:

**Theorem (Z.Lin, B-gravejat-Saut-Smets )**

Let $c \in (-\sqrt{2}, \sqrt{2})$. Given any $\varepsilon > 0$, there exists some $\delta > 0$ such that, if

$$d(\Psi^0, U_c) \leq \delta,$$

then

$$\sup_{t \in \mathbb{R}} \inf_{(a, \theta) \in \mathbb{R}^2} d(\Psi(\cdot, t), e^{i\theta} U_c(\cdot - a)) \leq \varepsilon.$$  

The energy space $\mathcal{X}(\mathbb{R})$ with corresponding metric space structure is

$$\mathcal{X}(\mathbb{R}) = \{ \psi : \mathbb{R} \to \mathbb{C}, \text{ s.t. } \partial_x \psi \in L^2(\mathbb{R}) \text{ and } 1 - |\psi|^2 \in L^2(\mathbb{R}) \}\,$$

The proof relies essentially of minimizing properties of the soliton.
Asymptotic stability in 1D

Let \( c \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\} \).

**Theorem (B-Gravejat-Smets 2013)**

There exists \( \delta^* > 0 \) s.t. if \( \Psi^0 \in \mathcal{X}(\mathbb{R}) \) and

\[
d_\mathcal{X}(\Psi^0, U_c) < \delta^*,
\]

then there exist \( c^* \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}, \ a \in C^1(\mathbb{R}, \mathbb{R}) \) and \( \theta \in C^1(\mathbb{R}, \mathbb{R}) \) s.t. the solution \( \Psi \) to (GP) with initial datum \( \Psi^0 \) satisfies as \( t \to +\infty \)

\[
\begin{cases}
  e^{-i\theta(t)}\Psi(\cdot + a(t), t) \to U_{c^*} \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbb{R}), \\
  a'(t) \to c^*, \quad \text{and} \quad \theta'(t) \to 0,
\end{cases}
\]

The proof requires **dispersive properties** of the equation, and is inspired by earlier works of Martel and Merle on KdV.
Some ideas in the proof

We write \( \Psi = \rho e^{i\varphi} \) and introduce the functions \( \eta := 1 - \rho^2 \) and \( \nu := -\partial_x \varphi \). The Ginzburg-Landau energy becomes

\[
E(\eta, \nu) = \frac{1}{8} \int_{\mathbb{R}} \frac{(\partial_x \eta)^2}{1 - \eta} + \frac{1}{2} \int_{\mathbb{R}} (1 - \eta) \nu^2 + \frac{1}{4} \int_{\mathbb{R}} \eta^2,
\]

and the momentum \( P \) writes

\[
P(\eta, \nu) = \frac{1}{2} \int_{\mathbb{R}} \eta \nu
\]

The conservation law for the momentum is the central tool:

\[
\partial_t (\eta \nu) = -\partial_x \left( (1 - 2\eta) \nu^2 + \frac{\eta^2}{2} + \frac{(3 - 2\eta)(\partial_x \eta)^2}{4(1 - \eta)^2} \right) - \frac{1}{2} \partial_{xxx} (\eta + \ln(1 - \eta))
\]

positive for small \( \eta \)

negative for small \( \eta \)
For $c \neq 0$ the soliton $U_c = \rho_c e^{i\varphi_c}$ is represented by the pair

$$Q_c := (\eta_c, v_c),$$

We decompose a solution near a soliton as

$$(\eta(\cdot + a(t), t), v(\cdot + a(t), t)) = Q_c(t) + \varepsilon(\cdot, t),$$

this decomposition is **unique** if we impose the orthogonality conditions

$$\langle \varepsilon(\cdot, t), \partial_x Q_c(t) \rangle_{H^1 \times L^2} = \langle \varepsilon(\cdot, t), P'(Q_c(t)) \rangle_{H^1 \times L^2} = 0.$$
An integral form of conservation of momentum

We choose as test function in the conservation law for $P$ the function

$$\Phi(x) := \frac{1}{2}(1 + \tanh(\nu_c x)) \geq 0 \text{ where } \nu_c := \frac{\sqrt{2 - c^2}}{8}$$

We define the momentum to the right of the soliton by

$$I_R(t) \equiv \frac{1}{2} \int_{\mathbb{R}} \eta v(x + a(t), t) \Phi(x - R) dx \text{ for any } R \in \mathbb{R}.$$
Monotonicity at infinity

Lemma

There exists $A_c > 0$ such that

$$\frac{d}{dt} \left[ I_R(t) \right] \geq \frac{1}{A_c} \int_{\mathbb{R}} \left[ (\partial_x \eta)^2 + \eta^2 + \nu^2 \right] (x + a(t), t) \Phi'(x - R) \, dx$$

$$-A_c e^{-2\nu_c |R|},$$

for any $t \in \mathbb{R}$. 

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Monotonicity on the soliton

The pair $\varepsilon$ satisfies

$$\partial_t \varepsilon = J\mathcal{H}_c \varepsilon + J\mathcal{R}_c \varepsilon + (a' - c)(\partial_x Q_c + \partial_x \varepsilon) - c' \partial_c Q_c,$$

where

$$J = \begin{pmatrix} 0 & -2\partial_x \\ -2\partial_x & 0 \end{pmatrix} = -2S \partial_x,$$

$$\mathcal{H}_c = E''(Q_c) - cP''(Q_c),$$

and

$$\mathcal{R}_c \varepsilon := E'(Q_c + \varepsilon) - E'(Q_c) - E''(Q_c)(\varepsilon).$$
We introduce a dual variable $u$ (as in Martel-Merle)

$$u = S\mathcal{H}_c \varepsilon.$$

The linear part of the equation for $u$ is given by

$$\partial_t u = -2S\mathcal{H}_c \partial_x u + \text{h.o.t.},$$

so that

$$\frac{d}{dt} \langle Mu, u \rangle_{L^2(\mathbb{R})^2} = -4\langle SMu, \mathcal{H}_c \partial_x u \rangle_{L^2(\mathbb{R})^2} + \text{h.o.t.},$$

for any symmetric matrix function $M \in C^\infty(\mathbb{R}, S_2(\mathbb{R}))$.

**Proposition**

*There exists a number $K > 0$ such that*

$$\frac{d}{dt} \langle M_c u, u \rangle_{L^2(\mathbb{R})^2} \geq \frac{1}{K} \int_{\mathbb{R}} e^{-\sqrt{2}|x|} \left[ (\partial_x u_1)^2 + (u_1)^2 + (u_2)^2 \right](x,t) dx$$

$$- K \| \varepsilon(\cdot, t) \|_{H^1 \times L^2} \| u(\cdot, t) \|_{H^1 \times L^2}^2,$$

*for any* $t \in \mathbb{R}$.  

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Asymptotic stability in higher dimensions

This question remains essentially open, at least concerning rigorous proofs.

For instance, one may investigate the asymptotic stability of the stationary vortex. Despite the fact that its energy is infinite, a Cauchy theory (B-Jerrard-Smets) is available which allows to address the issue.

Remark

- The modulation part might be simpler then in the 1D-case, since the vortex does not belong to a branch of special solutions.
- The hydrodynamical formulation is difficult to implement, since energy estimates do not yield $L^\infty$ bounds.
The question is the following:
Given an initial datum close to the vacuum, that is such that
\[ \rho \approx 1. \]

Can it generate vorticity?

**Conjecture:** YES at least in 3D, in particular starting near the unstable branch of traveling waves.

No definite answer yet to the question: only estimates for a lower bound on time when vorticity appears.
Long waves asymptotics

Recall that (HGP) contains the additional quantum pressure term

$$\varrho \nabla \tilde{p} = \varrho \nabla \left( \frac{\Delta \rho}{\rho} \right)$$

involves third order derivatives.

At least formally, one may neglect this term in a long wave asymptotics, which means considering solution $\Psi_\varepsilon$ of the form

$$\begin{cases}
\rho_\varepsilon^2(x) = 1 + a_\varepsilon(\varepsilon x) \\
v_\varepsilon(x) = u_\varepsilon(\varepsilon x),
\end{cases}$$

where

- $\Psi = \rho^2 \exp i\varphi$, $v = 2\nabla \varphi$.
- $\varepsilon > 0$ denotes a small parameter possibly tending to 0
- suitable bounds are imposed on $a_\varepsilon$ et $u_\varepsilon$. 

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Theorem (B-Danchin-Smets. 2010)

Let \( s > 1 + \frac{N}{2} \). There exists a constant \( C \equiv C(s, N) \) such that for any initial datum \( (a_\varepsilon^0, u_\varepsilon^0) \) satisfying the condition

\[
C_\varepsilon \| (a_\varepsilon^0, u_\varepsilon^0) \|_{H^{s+1} \times H^s} \leq 1
\]

there exists a time \( T_\varepsilon \geq \frac{1}{C_\varepsilon^2 \| (a_\varepsilon^0, u_\varepsilon^0) \|_{H^{s+1} \times H^s}} \) such that the system \((HGP)\) has a unique solution \((a_\varepsilon, u_\varepsilon)\) such that

\[
\frac{1}{2} \leq \rho (\cdot, \varepsilon, t) \leq 2 \quad \text{for} \quad t \in [0, T_\varepsilon]
\]

Moreover

\[
\| (a_\varepsilon (\cdot, t), u_\varepsilon (\cdot, t)) \|_{H^{s+1} \times H^s} \leq C \| (a_\varepsilon^0, u_\varepsilon^0) \|_{H^{s+1} \times H^s}
\]
Recall that the **compressible Euler system** is a **symetrizable hyperbolic system**, and therefore the **energy method** can be used to establish existence at least on some time interval.

We adapt the **method** to our setting. An important difficulty is to handle
- the **dispersive term** (quantum pressure), which is of third order
- in particular to perform the various integration by parts

the dispersive term, which is of third order
Remarks

- A typical initial datum we have in mind is

\[
\psi^0(x) = \sqrt{1 + \frac{\varepsilon}{\sqrt{2}}} a^0(\varepsilon x) \exp(i \varphi^0(\varepsilon x)),
\]

where \( u^0 \equiv 2\nabla \varphi^0 \) and \( a^0 \) do not depend on \( \varepsilon \) and belong to \( H^{s+1} \times H^s \). On then get the lower bounds

\[
T_\varepsilon \geq \frac{c}{\varepsilon^2}
\]

- One may also consider data of the form \( a^0_\varepsilon = \varepsilon^{-1} a^0 \) and \( u^0_\varepsilon = \varepsilon^{-1} u^0 \) to recover the Euler regime.
Linearization near vacuum

For densities \( \rho \) close to 1 and small vector fields \( \nu \), one may neglect at least formally the nonlinear terms, so that, as it is well-known that the compressible Euler equation reduces to the linear wave equation

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t \rho + \text{div}(\rho \nu) = 0, \\
\rho(\partial_t \nu + \nu \cdot \nabla \nu) + \nabla \rho^2 = 0.
\end{array} \right.
\end{align*}
\]

\[\downarrow\]

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t \rho + \text{div} \nu = 0, \\
\partial_t \nu + 2\nabla \rho = 0.
\end{array} \right.
\end{align*}
\]

The propagation speed is given by \( c_s \)

\[
c_s = \sqrt{\frac{\partial p(\rho)}{\partial \rho} \bigg|_{\rho=1}} = \sqrt{\frac{\partial \rho^2}{\partial \rho} \bigg|_{\rho=1}} = \sqrt{2}.
\]
The **very same principle** applies directly to (GP) in the long-wave limit ondes longues, for small data of the form

\[
\begin{align*}
\rho_\varepsilon^2(x, t) &= 1 + \frac{\varepsilon}{\sqrt{2}} a_\varepsilon(\varepsilon x, \varepsilon t), \\
v_\varepsilon(x, t) &= \varepsilon u_\varepsilon(\varepsilon x, \varepsilon t).
\end{align*}
\]

Here \( \varepsilon \) tends asymptotically to 0 and where the functions \( a_\varepsilon \) and \( u_\varepsilon \) are suitably bounded. The functions \( \rho_\varepsilon^2(x, t) - 1 \) and \( v_\varepsilon \) are hence \( O(\varepsilon) \) instead of \( O(1) \) in the ”Euler” regime.
with these notations and accelerating times by $\varepsilon^{-1}$, the system $5HGP$) writes

\[
\begin{align*}
\partial_t a_\varepsilon + \sqrt{2} \text{div} u_\varepsilon &= -\varepsilon \text{div}(a_\varepsilon u_\varepsilon), \\
\partial_t u_\varepsilon + \sqrt{2} \nabla a_\varepsilon &= \varepsilon \left( -u_\varepsilon \cdot \nabla u_\varepsilon + 2 \nabla \left( \frac{\Delta \sqrt{2} + \varepsilon a_\varepsilon}{\sqrt{2} + \varepsilon a_\varepsilon} \right) \right).
\end{align*}
\]  

(LHGP)

We recover on the r. h.s the linear wave operator with speed $c_s = \sqrt{2}$ (consistency). The l.h.s is formally small since there is a factor $\varepsilon$. It contains third order derivatives related to the dispersive nature of $(GP)$, which is indeed a Schrödinger equation.
convergence towards the wave equation

The previous a priori upper bound together with consistency yields the main point in order to approach (GP) by the wave equation.

Corollary

Let $s$, $a_\varepsilon^0$ and $u_\varepsilon^0$ be as in the previous theorem and let $(a, u)$ be the solution to the free wave equation

\[
\begin{cases}
\partial_t a + \sqrt{2} \operatorname{div} u = 0 \\
\partial_t u + \sqrt{2} \nabla a = 0,
\end{cases}
\]

with initial data $(a_\varepsilon^0, u_\varepsilon^0)$. If $\varepsilon \leq 1$ and $0 \leq t \leq T_\varepsilon$ then we have

\[
\|(a_\varepsilon, u_\varepsilon)(t) - (a, u)(t)\|_{H^{s-2}} \leq C \left[ \varepsilon t \|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s}^2 + \varepsilon^2 t \|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s} \right].
\]

Notice in particular the loss of three orders of derivatives related to the quantum pressure.

Fabrice Bethuel
On the dynamics of the Gross-Pitaevskii equation
In dimension $N \geq 2$, we use the dispersive properties of the linearized operator to improve the result: it is given near $(0, 0)$ by

$$L_\varepsilon(a, u) = \left( \partial_t a + \sqrt{2} \text{div} u, \partial_t u + \sqrt{2} \nabla a - \sqrt{2} \varepsilon^2 \nabla \Delta a \right),$$

Its dispersive properties are better than the ones of the wave operator. Its Fourier transform is given for $\xi \in \mathbb{R}^N$ and $t \in \mathbb{R}$ by

$$L_\varepsilon(\hat{a}, u)(\xi, t) = \left( \partial_t \hat{a}(\xi, t), \partial_t \hat{u}(\xi, t) \right) + i \begin{pmatrix} 0 & \sqrt{2} \xi^T \\ \sqrt{2} + \sqrt{2} \varepsilon^2 |\xi|^2 & 0 \end{pmatrix} \begin{pmatrix} a(\xi, t) \\ u(\xi, t) \end{pmatrix}.$$
Deriving Strichartz type estimates for $L_\epsilon$ we obtain

**Theorem**

Assume that $s > 2 + \frac{N}{2}$. Then we have for the the time $T_\epsilon$ the improved lower-bounds

\[
T_\epsilon \geq c_\epsilon^{-\frac{7}{3}} \left\| (a_\epsilon^0, u_\epsilon^0) \right\|_{H^{s+1} \times H^s}^{-\frac{4}{3}} \quad \text{if } N = 2.
\]

\[
T_\epsilon \geq c_\alpha \epsilon^{-2-\alpha} \left\| (a_\epsilon^0, u_\epsilon^0) \right\|_{H^{s+1} \times H^s}^{-1-\alpha} \quad \text{if } N = 3 \text{ and } 0 < \alpha < 1,
\]

\[
T_\epsilon \geq c_\epsilon^{-3} \left\| (a_\epsilon^0, u_\epsilon^0) \right\|_{H^{s+1} \times H^s}^{-2} \quad \text{if } N \geq 4,
\]

We are then able to compare as before the solution with that of the linearized problem on the corresponding interval.
It may be worthwhile to compare the existence results with the corresponding ones for the irrotational compressible Euler equation with smooth compactly supported perturbations of size or order $\varepsilon$ of a constant state. In that case, the corresponding $T_{\varepsilon}$ is known to be:

- $T_{\varepsilon} = +\infty$ for $N \geq 4$
- $T_{\varepsilon} \geq \exp\left(\frac{c}{\varepsilon}\right)$ for $N = 3$
- $T_{\varepsilon} \geq c\varepsilon^{-2}$ for $N = 2$
- $T_{\varepsilon} \geq c\varepsilon^{-1}$ for $N = 1$.

[Sideris, Klainerman]