RESEARCH ARTICLE

Using network coding to achieve the capacity of deterministic relay networks with relay messages

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ABSTRACT

In this paper, we derive the capacity of the deterministic relay networks with relay messages. We consider a network that consists of five nodes, four of which can only communicate via the fifth one. However, the fifth node is not merely a relay as it may exchange private messages with the other network nodes. First, we develop an upper bound on the capacity region based on the notion of a single-sided genie. In the course of the achievability proof, we also derive the deterministic capacity of a four-user relay network (without private messages at the relay). The capacity achieving schemes use a combination of two network coding techniques: the simple ordering scheme and detour scheme. In the simple ordering scheme, we order the transmitted bits at each user such that the bi-directional messages will be received at the same channel level at the relay, while the basic idea behind the detour scheme is that some parts of the message follow an indirect path to their respective destinations. This paper, therefore, serves to show that user cooperation and network coding can enhance throughput, even when the users are not directly connected to each other. Finally, we make a conjecture about the capacity region of the general \(K\)-node relay network with relay messages. Copyright © 2016 John Wiley & Sons, Ltd.

KEYWORDS

network coding; deterministic relay networks; capacity of relay networks; bi-directional relay networks

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1. INTRODUCTION

Cooperative communications for wireless networks have gained much interest, because of its potential in improving the performance of wireless networks. The relay network, where an additional node acting as a relay is supporting the exchange of information between the network users, is an important building block for future wireless communications. An extensive research has been conducted to study the fundamental limits of this cooperation (e.g. [1–10]).

The capacity of the Gaussian relay network is still an elusive goal even for the simplest setup, that is, a single-source single-destination topology. However, Avestimehr, Diggavi and Tse presented the deterministic channel model, through which we can get approximate results for the capacity of the Gaussian relay networks [5]. The Deterministic channel model captures the two main features of wireless communication: broadcasting and superposition of different signals. By eliminating the effect of noise, it allows us to focus only on the interactions between the different signals. In other words, the deterministic channel model gives a very close approximation of the Gaussian network when both the signal-to-noise ratio (SNR) and signal-to-interference ratio are high. Therefore, the insights gleaned from the deterministic model can be used to find approximations of the capacity for the Gaussian channels. The deterministic model has been used to study the capacity of different network topologies.

1.1. Related work

The work performed by authors in [6] investigated the capacity region of the deterministic multi-pair bidirectional relay network, which is a generalisation of the bidirectional relay channel. They proposed a simple equation-forwarding strategy that achieves this capacity region, which is tight to the cut-set upper bound, in which different pairs are orthogonalized on the signal level space and...
the relay just re-orders the received equations created from the superposition of the transmitted signals on the wireless medium and forwards them. We call this scheme the simple ordering scheme (SOS). Then, from the insights of this work, the authors of [7] used a combination of lattice codes and random Gaussian codes at the source nodes to propose a coding scheme that achieves to within 2 bits per user of the cut-set upper bound on the capacity of the two-pair two-way relay network.

Reference [8] studied the two-way X-Channel. First, a new upper bound based on the notion of a single-sided genie was developed, then it was used to characterize the deterministic multicast capacity of their network. To prove the achievability, the authors proposed the idea of Detour Schemes that route some bits intended for a certain receiver via alternative paths when they cannot be accommodated on direct routes. Thereafter, the capacity of the deterministic Y-channel was defined in [9]. Building on this result, the authors investigated the capacity of the Gaussian Y-channel in [10]. Then, we extended these works in [11], by considering a four-user relay network with no direct link, where each user wishes to exchange a number of private messages with the other three users via the relay node. Achievability of the capacity region was demonstrated via two detour schemes (DS), which are different from the ones used in [8] because of the different nature of the multicast network. After that, in [12], we studied the capacity of a three-user relay network where the relay node is interested in exchanging some private messages with the other network users. It was shown that, if all messages emanating from the ‘relay node’ are transmitted first, we obtain a reduced capacity region in the form of the capacity region of an asymmetric three-user relay network, the achievability of which was proven using a combination of the SOS and a DS. The authors in [13] considered the Gaussian case of this network topology and studied its capacity region.

Investigating the degrees of freedom of the relay network has garnered a huge research interest, because it provides another performance metric that is easier to be characterised compared with the capacity. (e.g. [14–17]). Recently, we applied some of the ideas, developed for the deterministic relay networks, in multiple-input multiple-output (MIMO) Gaussian set-up in [18].

1.2. Considered scenarios and contributions

It can be observed from [8,9,11,12] that the achievability technique differs from one topology to another depending on the number of users; thus, a question rises, how can we achieve the capacity region of the K-user relay network. To answer this question, we consider a deterministic four-user multicast relay network with no direct links, where each user can exchange private messages with the other network nodes. Additionally, the relay is interested in exchanging some private messages with the network users. This situation resembles the case where a base station relays messages between users and delivers messages from the backbone system to the end users as well. Furthermore, this model may represent a femtocell with intra-messages and inter-messages. A distinguishing feature of this work is the assumption of non-reciprocal channels. In fact, one can see this work as a generalisation of the work in [11] and [12]. In contrast to [11], it considers four-user relay network with private messages and non-reciprocal channels, and in contrast to [12], it considers an extra user in the network. Therefore, results in [8,9,11,12] can be obtained from the results presented here with appropriate settings of system parameters.

We start by developing a new upper bound on the capacity region based on the notion of single-sided genie. Then, we prove the achievability of this upper bound in two steps. First, relay private messages are delivered to their intended recipients. After removing the delivered messages from the network, we derive and achieve the capacity region of the resulting asymmetric four-user relay network. This capacity is achieved by using one of the two schemes: either the SOS or the DS. The results presented in this paper can be considered as an initial step towards characterising the capacity of the Gaussian counterpart of the considered model. Note that from our achievability proof, we only need to detour the bits over multiples of three-node cycles, in spite of the fact that one can form cycles that contain more than three nodes, that is, four-node cycles. This results in a constant delay in data delivery that does not exceed one time slot as we explain in Subsection 5.2. At a more fundamental level, this paper serves to show that network coding, whether through relaying messages for other users or through aligning interference at the relay has the potential to greatly enhance network throughput. Also, it is worth mentioning that we had showed the role of using DS in achieving the degrees of freedom region of the MIMO relay networks in [18], which indicates that the techniques developed in this paper may also serve to achieve the degrees of freedom region of the MIMO case of the considered model. Finally, we make a conjecture about the capacity region of the K-node relay network with relay messages.

1.3. Outline

In the following section, we describe the system model. In Section 3, we state our main result, which is the capacity region of the four-user multicast relay network with private messages from the relay. In Section 4, we present the first part of our achievability proof, which results in a reduced network in the form of a four-user non-reciprocal multicast relay network. The capacity region of this reduced network and its achievability are studied in Section 5. Our achievability techniques are illustrated using numerical examples in Section 6. The development of the upper bound on the capacity region based on the notion of single-sided genie is explained in Section 7. In Section 8, some comments regarding the generalisation to the K-node network are presented. Finally, we summarise our conclusions in Section 9.
2. SYSTEM MODEL

Consider a network consisting of five nodes as shown in Figure 1. Each node aims to exchange private messages with the other four nodes. The nodes from 1 to 4 have no direct links between them; thus, they can only communicate via node 5. However, node 5 is not merely a relay, as it exchanges private messages with the other four nodes. The message from node \( i \) to \( j \) has a rate given by \( R_{ij} \). Using the deterministic channel model [5], we denote the channel gain from node \( i \) to node \( j \) by \( n_{ij} \). Therefore, we assume the non-reciprocal scenario, where the \( n_{ij} \neq n_{ji} \).

Therefore, we assume that the uplink channel gains satisfy \( n_{5j} \geq n_{55} \geq n_{w5} \), while the downlink channel gains satisfy \( n_{5d} \geq n_{5b} \geq n_{5c} \geq n_{5d} \), where \( \{a, b, c, d\} \in \{1, 2, 3, 4\} \). Table I summarizes the important notations used throughout this paper.

3. MAIN RESULT

In this section, we present the main result of this work, which is the exact characterisation of the capacity region of the considered network.

Theorem 1. The capacity region of the four-user multicast relay network with relay private messages is given by the integral rate tuples that satisfies the inequalities (1)–(14).

\[
R_{n5} + R_{w5} + R_{w5} \leq n_{w5} \tag{1}
\]

\[
R_{n5} + R_{n5} + R_{w5} + R_{w5} + R_{w5} + R_{w5} + \max(R_{w5}, R_{w5}) \leq n_{w5} \tag{2}
\]

\[
R_{n5} + R_{n5} + R_{n5} + R_{w5} + R_{w5} + \max \left\{ \frac{[R_{w5} + R_{w5} + \max(R_{w5}, R_{w5})]}{[R_{w5} + R_{w5} + \max(R_{w5}, R_{w5})]} \right\} \leq n_{w5} \tag{3}
\]

\[
R_{n5} + R_{n5} + R_{n5} + R_{w5} + R_{w5} \leq n_{w5} \tag{4}
\]

\[
R_{n5} + R_{n5} + R_{w5} + R_{w5} \leq n_{w5} \tag{5}
\]

\[
R_{n5} + R_{n5} + R_{w5} + R_{w5} \leq n_{w5} \tag{6}
\]

\[
R_{n5} + R_{n5} + R_{n5} + R_{w5} + R_{w5} \leq n_{w5} \tag{7}
\]

\[
R_{5d} + R_{ad} + R_{bd} \leq n_{5d} \tag{8}
\]

\[
R_{5c} + R_{5d} + R_{ad} + R_{bd} + R_{ac} + \max(R_{cd}, R_{dc}) \leq n_{5c} \tag{9}
\]

\[
R_{5b} + R_{5c} + R_{5d} + R_{ad} + R_{ac} + \max(R_{cd}, R_{dc}) \leq n_{5b} \tag{10}
\]
where $R_{5X} = R_{5} + R_{a5} + R_{5} + R_{w5}$, and $R_{5X} = R_{5a} + R_{5b} + R_{5c} + R_{5d}$.

The aforementioned theorem indicates that all rate combinations represented by the above inequalities are achievable by our proposed schemes, the proof of this part is detailed in Sections 4 and 5. Furthermore, the network cannot support any rate combination that lies outside the defined region in Theorem 1. For instance, condition (1) ensures that the sum of the rates transmitted by node $w$ cannot exceed the capacity of the out link, i.e., $n_{w5}$. This is illustrated in details in Section 7. Because these two aspects, namely, the achievability and the converse, are derived using different techniques, reference to (1)–(14) will be made in the aforementioned sections.

4. ACHIEVABILITY

In this section, we prove the achievability of all integral rate tuples that satisfy the conditions stated in Theorem 1. As a first step to our achievability scheme, we serve the uplink and downlink messages of node 5, which are represented by the rates $R_{5} \text{ and } R_{5}$, respectively. Interestingly, the network with the remaining rates is reduced to an asymmetric four-user relay network, as illustrated in Appendix A. To complete the achievability proof, we derive the capacity region of this reduced network in Section 5 and show that the reduced rate tuples are achievable.

In the uplink phase, assign the indices $\ell_1$ to $\ell_{n5}$ to the channel levels, from lowest to highest, such that node $i$ cannot send bits on levels higher than $\ell_{ni}$. Therefore, starting by node $i$, we assign levels $\ell_{ni}$ through $\ell_{n5} - R_{5}$ to the message from node $i$ to node 5. Then, starting from level $n_{w5}$ or $n_{w5} - [R_{5} - (n_{w5} - n_{w5})] + 1$, whichever is smaller, we assign levels for the message $R_{5a}$ and so on. A similar operation is performed during the done link phase. This is illustrated by several examples in Figures 2 and 3. This procedure of assigning the channel levels for node 5 messages is equivalent to subtracting the rates related to node 5, that is, ($R_{5}$ and $R_{5}$), from both sides of all inequalities stated in Theorem 1. As detailed in Appendix A, this results in the reduced capacity region expressed by (15)–(28) in Theorem 2, where the reduced capacity gains are given by

$$R_{5x} + R_{ba} + R_{ca} + R_{da} + \max \left\{ R_{cb} + R_{db} + \max (R_{cd}, R_{dc}), [R_{bc} + R_{dc} + \max (R_{bd}, R_{db})], [R_{bd} + R_{cd} + \max (R_{bc}, R_{cb})] \right\} \leq n_{5x} \quad (11)$$

$$R_{5x} + R_{ab} + R_{cb} + R_{db} + \max \left\{ R_{ca} + R_{da} + \max (R_{cd}, R_{dc}), [R_{ad} + R_{cd} + \max (R_{ac}, R_{ca})], [R_{cd} + R_{ad} + \max (R_{ac}, R_{ca})] \right\} \leq n_{5a} \quad (12)$$

$$R_{5x} + R_{ac} + R_{bc} + R_{dc} + \max \left\{ R_{ba} + R_{da} + \max (R_{bd}, R_{db}), [R_{ab} + R_{db} + \max (R_{ad}, R_{da})], [R_{db} + R_{ad} + \max (R_{ab}, R_{ba})] \right\} \leq n_{5a} \quad (13)$$

$$R_{5x} + R_{ad} + R_{bd} + R_{cd} + \max \left\{ R_{ba} + R_{ca} + \max (R_{bc}, R_{cb}), [R_{ab} + R_{cb} + \max (R_{ac}, R_{ca})], [R_{ac} + R_{bc} + \max (R_{ad}, R_{bd})] \right\} \leq n_{5a} \quad (14)$$

Figure 2. Assigning levels in uplink phase: (a) $R_{5a} < n_{5a} - n_{5b}$, $R_{5d} < n_{5b} - n_{5c}$ and $R_{5c} < n_{5d} - n_{5c}$; (b) $n_{5a} - n_{5d} > R_{5a} > n_{5a} - n_{5c}$; (c) $n_{5b} > R_{5b} > n_{5d} - n_{5c}$; (d) $R_{5a} > n_{5a} - n_{5d}$.

Figure 3. Assigning levels in downlink phase: (a) $R_{5a} < n_{5a} - n_{5b}$, $R_{5b} < n_{5b} - n_{5c}$ and $R_{5c} < n_{5c} - n_{5d}$; (b) $n_{5a} - n_{5c} > R_{5a} > n_{5a} - n_{5b}$; (c) $n_{5b} > n_{5c} > R_{5b} > n_{5d} - n_{5c}$; (d) $R_{5a} > n_{5a} - n_{5d}$.
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Theorem 2, in the following sections.

5. ASYMMETRIC FOUR-USER RELAY NETWORKS

In this section, we study the capacity of the reduced network, which resulted after serving the messages related to node 5. Thus, we derive the capacity of the asymmetric four-user relay networks, which is given by the following theorem.

5.1. The capacity region of the asymmetric four-user relay networks

Theorem 2. The capacity of the deterministic four-user relay networks is given by the integral rate tuples that satisfy the inequalities (15)-(28).

\[
\begin{align*}
n_{Rd} &= n_{Rd} - \gamma R_{wu} - R_{uv} \leq n_{Wr} \\
R_{wt} + R_{wu} + R_{uv} \leq n_{Rd} \\
R_{wt} + R_{wu} + R_{uv} + \max \left\{ R_{wt} + \max (R_{uv} + R_{wu}), R_{wu} + \max (R_{uv} + R_{wu}) \right\} &\leq n_{Rd} \\
R_{ad} + R_{bd} + R_{cd} &\leq n_{Rd} \\
R_{db} + R_{ac} + R_{ad} + \max \left\{ R_{db} + \max (R_{ac} + R_{ad}), R_{ac} + \max (R_{db} + R_{ad}) \right\} &\leq n_{Rd}
\end{align*}
\]

where

\[
\beta = \begin{cases} 
\frac{\nu A - n_{s5} - n_{s6}}{\nu A} & R_{s5} \geq n_{s5} - n_{s6} \\
\frac{\nu A - n_{s5} - n_{s6}}{\nu A} & n_{s5} - n_{s6} \leq R_{s5} < n_{s5} - n_{s6} \\
\nu A & R_{s5} < n_{s5} - n_{s6}
\end{cases}
\]

\[
\gamma = \begin{cases} 
\frac{\nu B - n_{s5} - n_{s6}}{\nu B} & R_{s5} \geq n_{s5} - n_{s6} \\
\frac{\nu B - n_{s5} - n_{s6}}{\nu B} & n_{s5} - n_{s6} \leq R_{s5} < n_{s5} - n_{s6} \\
\nu B & R_{s5} < n_{s5} - n_{s6}
\end{cases}
\]

To continue our achievability proof of the original capacity region stated in Theorem 1, we need to prove the achievability of this reduced capacity region, stated in Theorem 2, in the following sections.
the remaining bits in each stream that were not included in lower segments.

5.2.2. SOS for the uplink. The XORed bits received in the uplink phase will be used 'as-is' in downlink phase. The relay needs only to reorder these bits to match the downlink segments described in the previous subsection.

**Lemma 1.** The SOS can achieve all the integral rate tuples in the intersection between the capacity region stated in Theorem 2 and the following extra conditions:

\[
\max\{(R_{wa} + R_{wb} + R_{wc}), (R_{wb} + R_{wc} + R_{wa})\} + R_{wb} + R_{wa} + R_{wc} \leq n_{Rd}
\]

\[
\max\{(R_{bc} + R_{cd} + R_{db}), (R_{bc} + R_{cd} + R_{db})\} + R_{db} + R_{bc} + R_{cd} \leq n_{Rd}
\]

\[
R_{ij} + R_{jk} + R_{ki} + \max\{(R_{ij} + R_{jk} + R_{ki}), (R_{ij} + R_{jk} + R_{ki})\} \leq n^*
\]

for any \(i, j, k \in \{1, 2, 3, 4\}\), where \(n^* = \min(n_{Ik}, n_{Rd})\).

**Proof.** See Appendix B.

5.2.2. The detour schemes.

Till now, we proved the achievability of the integral rate tuples that satisfy both the conditions in Theorem 2 and those in Lemma 1 simultaneously. By using the DS, we will prove the achievability of any integral rate tuple that satisfies the conditions in Theorem 2 but violates one or more from the conditions stated in Lemma 1. In essence, the DS converts the network into an equivalent one, where (29)–(32) are satisfied; thus, we can apply the SOS to this equivalent network.

Before explaining the details of our DS, we first observe that the set of extra conditions represented by (29)–(31) contain a three-node cycle represented by the data flow along the leading three terms in the left-hand side (LHS). In contrast, conditions represented by (32) contain two three-node cycles. For example, if \(\max(R_{ij}, R_{kj}) = R_{ji}\) and \(\max(R_{ik}, R_{kj}) = R_{ki}\), then the three-node cycles are \(i, j, k\) obtained from rates \(R_{ij}, R_{jk}\) and \(R_{ki}\), and the cycle \(i, j, k\)
obtained from rates $R_{ij}, R_{kl}$ and $R_{li}$. It is worth mentioning that the notion of these cycles will be important in defining our DS. Because any achievable rate tuple may violate more than one of the conditions expressed by (29)–(32), we define the maximum gap condition (MGC) as the condition having the maximum difference between the right-hand side and LHS of the inequalities over all violated conditions expressed by (29) and (32). According to the form of the MGC, we will use one of the two following DS.

5.2.2.1. Detour scheme 1 (DS 1). This scheme will be used when the MGC is in the form of (29), (30) or (31) for a certain $i,j,k,l$. In this case, the detour will be performed over the three-node cycle which exists in the MGC.

To simplify the notation, we assume without loss of generality that $(R_{ij} + R_{jl} + R_{ki}) \leq (R_{il} + R_{ij} + R_{ik})$; hence, the MGC in the form of (31) can be written as

$$R_{ij} + R_{jl} + R_{ki} + R_{il} + R_{ij} + R_{ik} > n^*$$

Now, we need to reduce the rates of the LHS by subtracting $\lambda$, such that

$$(R_{ij} + R_{jl} + R_{ki}) - \lambda + R_{il} + R_{ij} + R_{ik} \leq n^*$$

The subtracted $\lambda$-bits should be transmitted to their respective destinations via alternative paths (detours). Thus, all rates along this detour must be increased, while at the same time satisfying the other conditions in Theorem 2 and (29)–(32). For example, if we decide to detour $\lambda$-bits from the $R_{ij}$ via node $j$, this means each rate of $R_{ij}$ and $R_{lj}$ should be increased by $\lambda$. Therefore, whichever the rate we choose to detour this $\lambda$-bits from, the rates over the reverse cycle should be modified as

$$R_{ij} + R_{jl} + R_{ki} \rightarrow R_{ij} + R_{lk} + R_{jk} + 2\lambda$$

5.2.2.2. Detour Scheme 2 (DS 2). The MGC is in the form of (32) for a certain $i,j,k,l$. In this case, the detour will be performed through the two three-node cycles represented by the MGC, that is, $(k,l,j)$ obtained from rates $R_{ij}, R_{lj}$ and $R_{jk}$ and $(k,l,i)$ obtained from rates $R_{kl}, R_{jk}$ and $R_{ik}$. Again, we assume without loss of generality $\max(R_{ij}, R_{jl}) + \max(R_{ik}, R_{jl}) = R_{ik} + R_{jl}$; hence, the MGC in the form of (32) can be written as

$$R_{ij} + R_{jl} + R_{ki} + R_{ik} + R_{ij} \geq n^*$$

First, we identify the three-node cycles in the MGC, which are

$$R_{kl} \rightarrow R_{ij} \rightarrow R_{jl} \quad \text{and} \quad R_{kl} \rightarrow R_{il} \rightarrow R_{ik}$$

Again, we need to reduce the LHS by subtracting an integer $\alpha$ from the rates that represent the two cycles such that the reduced rates satisfy

$$R_{ij} + (R_{jk} + R_{dl} + R_{li} + R_{ik} + R_{ij}) - \alpha \leq n^* \quad (33)$$

The omitted $\alpha$-bits from the two cycles in (33) will be detoured over two cycles. For example, we may detour $\alpha$-bits from $R_{ij}$ via users $j$ and $l$; thus, the rates over this path will be modified as follows:

$$R_{ki} \rightarrow R_{kl} - \alpha, \quad R_{jl} \rightarrow R_{lj} + a_1, \quad R_{il} \rightarrow R_{il} + a_1, \quad R_{kl} \rightarrow R_{kl} + a_2, \quad \text{and} \quad R_{il} \rightarrow R_{il} + a_2$$

where $\alpha = a_1 + a_2$.

Therefore, whichever the rates we choose to detour some bits from, the rates over the reverse cycles should be increased as follows

$$R_{ik} + R_{kj} + R_{kl} + R_{il} \rightarrow R_{ik} + R_{kj} + R_{kl} + R_{il} + 2\alpha$$

Lemma 2. For all integer rate tuples for the four-node relay network satisfying Theorem 2 and where any of the conditions in Lemma 1 is violated, it is possible to modify the rates using one of the two DS to find an equivalent network, which can achieve the rate tuple via alternative paths.

Proof. See Appendix C.

Now, we have completed the achievability proof of any integral rate tuple that satisfies the conditions stated in Theorems 1 or 2. In the following section, our achievability techniques are illustrated numerically.

6. NUMERICAL RESULTS

In this section, we present two numerical examples that illustrate our achievability schemes.

Consider the five-node network channel gains are given as $N_{UL} = (n_{15}, n_{25}, n_{35}, n_{45})$, $N_{DL} = (n_{51}, n_{52}, n_{53}, n_{54})$ and a rate tuple $R = (R_{12}, R_{13}, R_{14}, R_{15}, R_{21}, R_{23}, R_{25}, R_{31}, R_{32}, R_{34}, R_{35}, R_{41}, R_{42}, R_{43}, R_{51}, R_{52}, R_{53}, R_{54})$, while for the reduced four-user relay network, the channel gains are given as $N_{UL}^R = (n_{1R}, n_{2R}, n_{3R}, n_{4R})$, $N_{DL}^R = (n_{1R}, n_{2R}, n_{3R}, n_{4R})$ and a rate tuple $R = (R_{12}, R_{13}, R_{14}, R_{15}, R_{21}, R_{23}, R_{25}, R_{31}, R_{32}, R_{34}, R_{35}, R_{41}, R_{42}, R_{43}, R_{51}, R_{52}, R_{53}, R_{54})$.

6.1. Example 1: the use of DS1

Consider $N_{UL} = (11, 5, 7, 1)$, $N_{DL} = (2, 8, 5, 11)$ and $R = (2, 0, 1, 0, 2, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1)$. After serving node 5 messages, we get a reduced network with $N_{UL}^R = (7, 4, 5, 0)$, $N_{DL}^R = (1, 5, 3, 7)$ and $R = (2, 0, 1, 0, 2, 1, 1, 1, 1, 0, 0, 0)$, which violates some of the conditions in Lemma 1, and the MGC is

$$R_{12} + R_{23} + R_{31} + R_{14} + R_{24} + R_{34} = n^* + \lambda = 7 + 1$$

We detour 1 bit over the cycle $1 \rightarrow 2 \rightarrow 3$. In particular, we will detour a bit from the rate $R_{12}$ via user 3, which results in the modified rate tuple $R = (1, 1, 1, 0, 2, 1, 1, 1, 1, 0, 0, 0)$. This modified rate tuple satisfies the conditions in Theorem 2 and Lemma 1; thus, we can apply the SOS, as illustrated in Figure 4.
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6.2. Example 2: the use of DS2
Consider $N_{UL} = (11, 10, 5, 3), N_{DL} = (3, 6, 10, 11)$ and $R = (2, 1, 0, 1, 0, 2, 1, 0, 0, 1, 1, 2, 0, 0, 1, 0, 2, 1, 1)$. First, we serve the messages related to node 5 in both uplink and downlink phases as illustrated in Section 4. Subsequently, we get a reduced network with $N'_{UL} = (8, 8, 3, 2), N'_{DL} = (3, 4, 7, 7)$ and $R' = (2, 1, 0, 0, 2, 1, 0, 0, 1, 2, 0, 0)$, which violates some of the conditions in Lemma 1, and the MGC is

$$R_{12} + R_{23} + R_{34} + R_{41} + R_{13} + R_{24} = n^* + \alpha = 7 + 2$$

Therefore, we will detour 1 bit from the rate $R_{24}$ via user 1, and 1 bit from the rate $R_{41}$ via user 3, which results in the modified rate tuple $R'' = (2, 1, 1, 1, 2, 0, 1, 0, 1, 1, 0, 1)$. This modified rate tuple satisfies the conditions in Theorem 2 and Lemma 1; thus, we can apply the SOS, as illustrated in Figure 5. In Figure 6, we show the performance of the proposed schemes. The SOS is able to achieve the combinations between $R_{13}$ and $R_{41}$ that lie on the blue line for the given rate tuple. By combining the DS with the SOS, we are able to achieve all the combinations that lie on the red line. Note that the red line represents the capacity region for this topology, that is, any rate combination above the red line is known to be not achievable by any coding technique.

7. THE UPPER BOUND BASED ON THE NOTION OF SINGLE-SIDED GENIE
In this section, we prove the converse parts of Theorems 1 and 2, that is, we show that the network cannot support any rate tuple that lies outside the defined regions, which implies the optimality of the proposed schemes. In the tra-
ditional cut-set bounds [19], network nodes are divided into two sets $S$ and $S'$, which represent the transmitting and receiving nodes, respectively. This bound follows from the max-flow min-cut theorem [19], that is, the sum of all transmission rates between any two sets of nodes cannot exceed the capacity of the communication links between them. For example, if we consider a set $S$ that contains only node 1, which has $n_{1,i} = 5$ and $S' = \{2,3,4,5\}$, then the sum of all communication rates transmitted from node 5 to 1 cannot exceed 5 as node 1 is able to receive a maximum of 5 distinct bits. In the considered set-up during the downlink phase, we assume that all nodes in $S'$, the receiving nodes, fully cooperate with each other and share all their side information. This type of cooperation is known as the two-sided genie aided bound, because it may be viewed as a genie exchanges the side information between the nodes on the same side of the cut. On the other hand, in the uplink phase, where all nodes transmit their signals to the relay, it makes two assumptions, the first is that a genie transfers the side information of all nodes in $S'$ to the relay. Therefore, the relay has more information than any other node in $S'$. This is clearly an upper bound, because if the relay failed to decode, then all other receiving nodes, with less amount of information, will also fail to decode. The second assumption is that all nodes in $S$ share all their side information. Applying the traditional cut-set bound to the relay network leads to loose bounds; therefore, a tighter single-sided genie aided upper bound was developed [8].

7.1. The downlink upper bound

If we consider the cut on the downlink phase with $S = \{i,j\}$ and $S' = \{k,l,5\}$, the two-sided genie, cut-set bound [19] is

$$R_{Si} + R_{Sj} + R_{ki} + R_{li} + R_{kj} + R_{lj} \leq \max(n_{Si}, n_{Sj})$$

(34)

However, let us assume that the genie transfers only all data of node $i$ to node $j$ except the data represented by $R_{ij}$. Now, node $j$ has more information than before; therefore, if it fails to decode $R_{ij}$, we are sure that it will fail to decode $R_{ij}$ without this additional information. Also, if node $j$ decoded $R_{ij}$ successfully, then it has its own side information in addition to all the side information of node $i$. Thus, if node $j$ fails to decode the remaining data, which is sent to it and node $i$, then we are sure that node $i$ will fail too. In the deterministic model, this is equivalent to

$$R_{Si} + R_{Sj} + R_{ki} + R_{li} + R_{kj} + R_{lj} + R_{ij} \leq \max(n_{Si}, n_{Sj})$$

(35)

Observe that the rate $R_{ij}$, which represents the information known to node $i$ and therefore the single-sided genie relays to node $j$, increases the LHS of (35) compared with the one of (34). This results in a tighter bound given by the one-sided genie.

Conversely, if the genie transfers only all the data of node $j$ to node $i$, the bound will be as follows:

$$R_{Si} + R_{Sj} + R_{ki} + R_{li} + R_{kj} + R_{lj} + R_{ij} \leq \max(n_{Si}, n_{Sj})$$

These two conditions can be combined as follows:

$$R_{Si} + R_{Sj} + R_{ki} + R_{li} + R_{kj} + R_{lj} + R_{ij} + \max(R_{ij}, R_{ji}) \leq \max(n_{Si}, n_{Sj})$$

We can notice that this is the form of condition (9).

For $S = \{i,j,k\}$ and $S' = \{l,5\}$, if we assume that the genie transfers all data from node $i$ to nodes $j$ and $k$, that is, $(R_{ij}$ and $R_{ik})$, and all data from node $j$ to node $k$, that is, $(R_{jk})$, therefore the data sent from node $k$ to nodes $i$ and $j$, that is, $(R_{ki}$ and $R_{kj})$, are not known at nodes $i$ and $j$, and the data sent from node $j$ to node $i$, that is, $(R_{ji})$, are not known at node $i$. This results in a tighter inequality as follows:

$$R_{Si} + R_{Sj} + R_{Si} + R_{li} + R_{lj} + R_{lk} + R_{ji} + R_{ki} + R_{kj} \leq \max(n_{Si}, n_{Sj}, n_{Sk})$$

It is clear that for the $S$, $S'$, the order in which the genie transfers the data will affect the resulting inequality. Therefore, the previous inequality represents only one of the different genie orders, namely, $i \rightarrow j \rightarrow k$, which must be taken into account to characterise the upper bound. Note that this is the cut that gives, through different genie orders, condition (10).

It should be mentioned that for $S = \{i\}$, the cut contains only one node and the single-sided genie bound coincides with the traditional two-sided genie. For example, $S = \{i\}$ and $S' = \{j,k,l,5\}$, we get

$$R_{Si} + R_{ij} + R_{ik} + R_{il} \leq n_{Si}$$

which gives us condition (8) in Theorem 1. In contrast, for the four-user relay network, we need the cut around the relay, where the one-sided genie bound depends only on the different genie orders. For example, if we assume the genie order $i \rightarrow j \rightarrow k \rightarrow l$, this means the genie transfers all data of node $i$ to nodes $j, k$ and $l$, then the data of node $j$ to nodes $k$ and $l$, and finally, it transfers the data from node $k$ to node $l$; therefore, the one-sided genie bound in this case will be

$$R_{li} + R_{lj} + R_{lk} + R_{ki} + R_{kj} + R_{lj} \leq n_{Ra}$$

which leads to conditions (25)–(28) in Theorem 2.

7.2. The uplink upper bound

For the uplink, let us take $S = \{i,j\}$ and $S' = \{k,l,5\}$, and the relay, that is, node 5 is only the receiving node, as no direct link is assumed between the other nodes. Now, we assume the one-sided genie from node $i$ to node $j$, that is, the genie transfers only $R_{ij}$ to the relay. In addition, we assume that the relay has all the side information of nodes $k$ and $l$. Therefore, if it fails, then nodes $k$ and $l$ will also fail. Now, the relay is in a better position to decode compared with node $i$, and it should be able to decode $R_{ij}$. In terms of the deterministic model, this is equivalent to

$$R_{Si} + R_{Sj} + R_{ik} + R_{il} + R_{jk} + R_{lj} + R_{ji} \leq \max(n_{Si}, n_{Sj})$$

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Again, if the genie order is exchanged, then the bound will be
\[ R_{S5} + R_{S\beta} + R_{i\delta} + R_{i\ell} + R_{ij} + R_{ij} \leq \text{max}(n_{i5}, n_{ij}) \]
These two conditions can be combined as follows:
\[ R_{S5} + R_{S\beta} + R_{i\delta} + R_{i\ell} + R_{ij} + R_{ij} + \text{max}(R_{ij}, R_{ij}) \leq \text{max}(n_{i5}, n_{ij}) \]
which gives us condition (2) in Theorem 1. For the cut set that contains two nodes, \( S = \{i, j, k\} \) and \( S' = \{l, 5\} \), for the genie order \( i \to j \to k \), we have the following bound:
\[ R_{S5} + R_{S\beta} + R_{i\delta} + R_{i\ell} + R_{ij} + R_{ij} + R_{ij} \leq \text{max}(n_{i5}, n_{ij}, n_{i5}) \]
which leads to conditions (3) in Theorem 1. Note that, for the asymmetric four-user relay network, again here we need the cut around the relay, where the bounds depend only on the genie order; therefore, from genie order \( i \to j \to k \) in uplink phase, we get a bound restricted to \( n_{iR} \), which has the same LHS of the bound we get from the reversed genie order in downlink phase, that is, \((l \to k \to j \to i)\) restricted to \( n_{R} \). Therefore, we can combine these resultant conditions, (18)–(21) and (25)–(28) in Theorem 2, to be restricted to \( n^* = \text{min}(n_{R}, n_{R}) \). By taking all cuts with all possible genie orders, we get the region stated in Theorems 1 and 2. However, some cuts do not add new constraints on the capacity region as explained in the following subsection.

### 7.3. Simplifying observations

First, we observe that we need not take into account all cuts where \( S \) contains node 5 with any other node, which significantly reduces the number of inequalities to consider. For example, let \( S = \{5, 1\} \) and let the genie transfer only the data from node 1 to node 5, that is, \( R_{15} \), this will provide the following bound:
\[ R_{S2} + R_{S3} + R_{S4} + R_{12} + R_{13} + R_{14} + R_{S5} \leq \text{max}(n_{i5}, n_{i5}, n_{53}, n_{54} + n_{15}) \]
This condition is implicitly satisfied from the resultant bounds from the cuts \( S = \{5\} \) and \( S = \{1\} \)
\[ R_{S1} + R_{S2} + R_{S3} + R_{S4} \leq \text{max}(n_{i5}, n_{i5}, n_{53}, n_{54}) \]
\[ R_{15} + R_{12} + R_{13} + R_{14} \leq n_{15} \]
Therefore, the cut \( S = \{5, 1\} \) does not result in a new constraint on the capacity region; thus, we omit it.

The second observation is that some cuts do not give a new constraint on the capacity region; therefore, they were not stated in Theorems 1 and 2. For example, the bound obtained from the cut \( S = \{v\} \) is implicitly included in the one obtained from the cut \( S = \{v, w\} \). For \( S = \{v\} \) and \( S = \{v, w\} \), we have
\[ R_{vw} + R_{vw} + R_{vt} \leq n_{vR} \]
\[ R_{vw} + R_{vt} + R_{vw} + R_{vt} \]
\[ + \text{max}(R_{vw}, R_{vt}) \]
\[ \leq \text{max}(n_{vR}, n_{vt}) = n_{vR} \]

Therefore, we do not need to take into account the cut \( S = \{v\} \) in characterising the capacity region stated in Theorem 2.

### 8. TOWARDS THE K-NODE RELAY NETWORK

From the insights obtained from this work, [11] and [12], we can make a conjecture about the capacity region of the K-node relay network with relay messages, where we have \( K - 1 \) users \( \{1, 2, \ldots, K - 1\} \), each of them wishes to exchange messages with the remaining network nodes via the K-th node, which acts as a relay beside its interest to transmit its private messages to each user.

#### 8.1. The upper bound

Based on the notion of one-sided genie, we can get the upper bound for the uplink phase as follows. For the cut-set that contains one node \( S = \{l\} \), we get
\[ \sum_{i=1, i \neq j}^{K} R_{ij} \leq n_{jk} \]  \( \text{(36)} \)

For the cut set that contains two nodes, \( S = \{j, k\} \):
\[ \sum_{i=1, i \neq j}^{K} R_{ij} + \sum_{i=1, i \neq j}^{K} R_{ik} \leq \text{max}(n_{jk}, n_{jk}) \]  \( \text{(37)} \)

For the cut set that contains three nodes, \( S = \{j, k, l\} \):
\[ \sum_{i=1, i \neq j}^{K} R_{ij} + \sum_{i=1, i \neq j}^{K} R_{ik} + \sum_{i=1, i \neq j}^{K} R_{il} \leq \text{max}(n_{jk}, n_{jk}, n_{jk}) \]  \( \text{(38)} \)

And so on, for the cut set that contains \( K - 1 \) users \( S = \{1, 2, \ldots, K - 1\} \):
\[ \sum_{i=1, i \neq j}^{K} R_{ij} + \sum_{i=1, i \neq j}^{K} R_{ik} + \sum_{i=1, i \neq j}^{K} R_{il} + \sum_{i=1, i \neq j}^{K} R_{ml} + \ldots \]
\[ + \sum_{i=1, i \neq j}^{K} R_{ej} \leq \text{max}(n_{jk}, n_{jk}, \ldots, n_{jk}) \]  \( \text{(39)} \)

for all \( \{j, k, l, \ldots, z\} \in \{1, 2, 3, \ldots, K\} \). We can proceed similarly to derive the upper bound on the downlink phase.

#### 8.2. Achievability

Again, after serving the K-th node messages, we obtain a reduced capacity region in the form of the one of an asym-
metric \((K - 1)-\)user relay network. The capacity of this reduced network is obtained using a combination of the SOS and DS. We can conjecture that the set of extra conditions needed to apply the SOS is cyclic conditions, that is, contain multiples of three-node cycle. If any of these cyclic conditions is violated, we will use the DS, we will focus on forcing the MGC to be satisfied by detouring a number of bits via all three-node cycles contained in this conditions. We tested our conjecture on five-user relay networks, and we found that the detour can be applied over multiples of three-node cycles, from one cycle up to four cycles, which support our conjecture.

9. CONCLUSIONS

In this paper, we characterised the capacity region of a deterministic four-user relay network, where the relay is interested in exchanging private messages with the network users. The use of a simplified, deterministic model allowed derivation of exact capacity results. However, the insights gained from this simplified model suggest that cooperation among users who are not directly connected, either by relaying other users’ messages (through detours) or by network coding, which took the form of XORing bits from different users in the SOS scheme, but could probably be through the use of Lattice codes in Gaussian channels, can further improve network throughput. We developed a new upper bound on the capacity region based on the notion of single-sided genie. After serving the messages related to the relay node, we obtained a reduced network in the form of the asymmetric four-user relay network. Thus, in the second part of our achievability argument, we proved the achievability of this reduced region via using the idea of the DS, where we sent some bits via alternative paths instead of sending them directly.

Because we considered a five-node network, this work strengthens the conjecture in [12]; that the capacity region \(K\)-node relay network with relay messages can be achieved in two steps: first, we serve the messages related to the relay, then we end with a capacity region of asymmetric \(K - 1\)-user relay network. The capacity of this reduced network can be achieved using combination of SOS and DS, where the detours are performed over multiple three-node cycles. Also, this work is a generalization for the work performed in [11] by considering a non-reciprocal scenario.

APPENDIX A: THE REDUCTION OF THE CAPACITY REGION

By applying the subtraction operation described in Section 4 on the conditions (4)–(7) and (11)–(14), we obtain the conditions (18)–(21) and (25)–(28), respectively. Now, we perform some mathematical simplifications to get a meaningful reduced region. By subtracting all rates related to node 5 from the both sides of conditions (2), (3) and (6), we obtain

\[
R_{wt} + R_{wu} + R_{vt} + R_{wu} + R_{uv} \leq n_5 - R_{v5} - R_{w5} \quad (A1)
\]

\[
R_{wt} + R_{wu} + R_{vt} + R_{wu} + R_{uv} + R_{wt} \leq n_5 - R_{v5} - R_{v5} - R_{w5} \quad (A2)
\]

\[
R_{wt} + R_{wu} + R_{vt} + R_{wu} + R_{uv} + R_{wt} \leq n_5 - R_{v5} - R_{v5} - R_{w5} \quad (A3)
\]

Because \(R_{wt} \geq 0\), then we have

\[
R_{vt} + R_{wt} + R_{wu} + R_{wu} + R_{wu} \leq n_5 - R_{v5} - R_{v5} - R_{w5} \quad (A4)
\]

\[
R_{wt} + R_{wu} + R_{vt} + R_{wu} + R_{wu} \leq n_5 - R_{v5} - R_{v5} - R_{w5} - R_{w0} \quad (A5)
\]

These conditions can be combined as

\[
R_{wt} + R_{wu} + R_{vt} + R_{vu} + R_{uv} \leq n_5 R \quad (A6)
\]

where

\[
n_R = \min\{n_5 - R_{v5} - R_{w5}, n_5 - R_{v5} - R_{v5} - R_{w5}, n_5 - R_{v5} - R_{v5} - R_{w5}, n_5 - R_{v5} - R_{v5} - R_{w5} - \beta\}
\]

Again, by applying the same process on conditions (2), (3) and (7), we get

\[
R_{wt} + R_{vu} + R_{vt} + R_{wu} + R_{uv} \leq n_5 R \quad (A7)
\]

Combining (A6) and (A7), we get (16).

Also, after subtracting the rates related to node 5 from the both sides of (3) and (5), we get

\[
R_{wt} + R_{vt} + R_{wu} + R_{vu} + R_{uv} \leq n_5 - R_{v5} - R_{v5} - R_{w5} \quad (A8)
\]

These two conditions can be combined as

\[
R_{wt} + R_{vt} + R_{wu} + R_{vu} + R_{uv} \leq n_R \quad (A9)
\]

where

\[
n_R = \min\{n_5 - R_{v5} - R_{v5} - R_{w5}, n_5 - R_{v5} - R_{v5} - R_{w5}, n_5 - R_{v5} - R_{v5} - R_{w5}, n_5 - R_{v5} - R_{v5} - R_{w5} - \beta\}
\]

Again, by subtracting the rates related to node 5 from the both sides of the conditions (3) and (5), we can get

\[
R_{wt} + R_{vt} + R_{wu} + R_{vu} + R_{uv} \leq n_R \quad (A10)
\]

The conditions (A8) and (A9) can be combined to get condition (17).

Also, by applying the same process on the conditions (1), (2), (3) and (7), we obtain the following conditions:

\[
R_{wt} + R_{wu} + R_{uv} \leq n_5 - R_{w5}
\]

\[
R_{wt} + R_{wu} + R_{vt} + R_{vu} + R_{uv} \leq n_5 - R_{v5} - R_{w5},
\]

\[
R_{wt} + R_{vt} + R_{wu} + R_{vu} + R_{uv} \leq n_5 - R_{v5} - R_{w5} - R_{w5},
\]

\[
R_{wt} + R_{wu} + R_{vu} + R_{vt} + R_{vu} + R_{uv} \leq n_5 - R_{v5} - R_{w5} - R_{w5} - R_{w5},
\]

Because any \(R_{ij} \geq 0\), we can get

\[
R_{wt} + R_{wu} + R_{uv} \leq n_5 - R_{v5} - R_{w5} \quad (A10)
\]
\( R_{wt} + R_{wu} + R_{ww} \leq n_{a5} - R_{a5} - R_{c5} - R_{w5}, \quad (A11) \)
\( R_{wt} + R_{wu} + R_{ww} \leq n_{s5} - R_{s5} \quad (A12) \)

The conditions (A10)–(A12) can be combined as
\[ R_{wt} + R_{wu} + R_{ww} \leq n_{wR} \quad (A13) \]
where
\[ n_{wR} = \min\{n_{s5} - R_{s5}, n_{w5} - R_{s5} - R_{w5}, n_{a5} - R_{a5} - R_{c5} - R_{w5}, n_{s5} - n_{s5}\} + \]
\[ = n_{w5} - R_{w5} - [\max(R_{s5} + R_{a5} + R_{c5} - (n_{s5} - n_{w5}), R_{a5} + R_{c5} - (n_{s5} - n_{w5}))] \]
which is the same as condition (15) in Theorem 2. Proceeding similarly for the downlink conditions, we obtain (22)–(28), where
\[ n_{Rb} = n_{s5} - R_{a5} - R_{s5} - R_{b5} - [R_{s5} - (n_{s5} - n_{s5})] + \]
\[ n_{Rc} = n_{s5} - R_{a5} - R_{s5} - \gamma \]
\[ n_{Rd} = n_{s5} - R_{a5} - R_{s5} - R_{d5}, \]
\[ n_{Rb} = n_{s5} - R_{a5} - R_{s5} - R_{b5} - [R_{s5} - (n_{s5} - n_{s5})] + \]
\[ = n_{s5} - R_{s5} - \max(R_{s5} + R_{b5} + R_{s5} - (n_{s5} - n_{s5}), R_{b5} + R_{s5} - (n_{s5} - n_{s5})). \]

Note that from the aforementioned expressions of the reduced channel gains, that is, \( n_{Rb} \) and \( n_{Rd} \), we can readily note that \( n_{R} \geq n_{R} \geq n_{R} \geq n_{R} \) and \( n_{R} \geq n_{R} \geq n_{R} \). Finally, we end up with the reduced capacity region, which is stated in Theorem 2. Interestingly, we can observe that this region coincides with the capacity region of asymmetric four-user relay network with channel gains \( n_{R}, n_{aR}, n_{R} \) and \( n_{R} \) in the uplink and \( n_{R}, n_{aR}, n_{R} \) and \( n_{R} \) in the downlink.

**APPENDIX B: PROOF OF LEMMA 1**

The SOS can only work if user \( i \) is able to transmit and receive all its data on the available number of levels \( n_{s5} \) and \( n_{s5} \), respectively, which means that each segment can be accommodated in the corresponding channel levels. The proof depends on finding the size of each of the four segments and applying this condition to it in both uplink and downlink phases.

For the uplink phase, the condition on the size of segment \( w \), which we call \( SS_{uw} \), is given by
\[ SS_{uw} = R_{wt} + R_{wu} + R_{ww} \leq n_{wR} \]
This condition is the same as condition (15) in Theorem 2.

Proceeding towards segment \( v \), we calculate \( SS_{uv} \), then apply the condition
\[ SS_{uv} + SS_{w} \leq n_{sR} \]
\[ \therefore R_{wt} + R_{wu} + R_{ww} + max(R_{vv}, R_{ww}) \leq n_{sR} \]
Again, this condition is the same as condition (16) in Theorem 2.

Then, we proceed towards segment \( u \); we calculate \( SS_{u} \), then apply the condition
\[ SS_{u} + SS_{v} + SS_{w} \leq n_{uR} \]
\[ \therefore R_{wt} + R_{wu} + R_{ww} + max(R_{uv}, R_{ww}) \]
\[ + max(R_{uv}, R_{uw}) \leq n_{uR} \]

By comparing this condition with the conditions (17), we can find that we need the following extra two conditions to guarantee that the aforementioned condition is satisfied:
\[ R_{wt} + R_{wu} + R_{uv} + R_{uv} + R_{wu} \leq n_{uR} \]
\[ R_{wt} + R_{wu} + R_{uv} + R_{uv} + R_{wu} \leq n_{uR} \]
Now, these two conditions can be combined to give condition (29) in Lemma 1.

Finally, we proceed towards segment \( t \); we calculate \( SS_{t} \), then apply the following condition:
\[ SS_{t} + SS_{u} + SS_{v} + SS_{w} \leq n_{tR} \]
\[ \therefore max(R_{tu}, R_{ut}) + max(R_{tv}, R_{vt}) + max(R_{tw}, R_{wt}) \]
\[ + max(R_{tw}, R_{uw}) + max(R_{tw}, R_{uv}) \leq n_{tR} \]

The aforementioned condition is equivalent to a combination of \( 2^5 = 64 \) conditions, depending on the rate achieving the maximum in each of the aforementioned terms; we are sure that 24 of them are already satisfied from the conditions (18)–(21). Thus, to apply the SOS, we should have rate tuple that satisfies the remaining 40 conditions; we found that these conditions can be written as (31) and (32) in Lemma 1, but restricted to \( n_{R} \). By following similar steps for the downlink phase, we obtain the conditions (31) and (32) with \( n^* = n_{R} \); thus, if we combined them with the conditions we get from the uplink phase, we get the conditions (31) and (32) as stated in Lemma 1, but this is time restricted to \( n^* = \min(n_{R}, n_{Rd}) \).

**APPENDIX C: PROOF OF LEMMA 2**

**C.1. Detour Scheme 1**

For simplicity, assume the MGC is in the form of (31) for \( \{i,j,k,l\} = \{1,2,3,4\} \) and \( \max\{R_{14} + R_{24} + R_{34}\} = R_{14} + R_{24} + R_{34} \), then the MGC is expressed as
\[ R_{12} + R_{23} + R_{31} + R_{14} + R_{24} + R_{34} = n^* + \lambda \quad (C1) \]

However, by combining the conditions (18)–(21) with (25)–(28), for any order of the channel gains, we get the following conditions:
By comparing these conditions with the MGC, we obtain
\[ R_{31} \geq R_{13} + \lambda, \quad R_{12} \geq R_{21} + \lambda, \quad R_{23} \geq R_{32} + \lambda \tag{C2} \]

Also, from the extra SOS conditions in Lemma 1, we have
\[ R_{12} + R_{23} + R_{31} + R_{41} + R_{24} + R_{34} \leq n^* + \lambda, \]
\[ R_{12} + R_{23} + R_{31} + R_{14} + R_{42} + R_{34} \leq n^* + \lambda, \]
\[ R_{12} + R_{23} + R_{31} + R_{14} + R_{24} + R_{43} \leq n^* + \lambda \]

Again, by comparing with the MGC, we get
\[ R_{14} \geq R_{41}, \quad R_{24} \geq R_{42}, \quad R_{34} \geq R_{43} \tag{C3} \]

Now, we need to choose the rate \( R_i \) from which we will subtract \( \lambda \) bits; thus, we have the following three options.

**C.1.1. Detour bits from \( R_{12} \).**

We can apply this detour as long as none of the following occurs: (i) \( d = 3 \) or \( w = 3 \); and (ii) \( c = 3 \) and \( d = 4 \), or \( v = 3 \) and \( w = 4 \). We apply the DS as follows:
\[ R_{12} \rightarrow R_{12} - \lambda, \quad R_{13} \rightarrow R_{13} + \lambda, \quad \text{and} \quad R_{32} \rightarrow R_{32} + \lambda \]

**C.1.2. Detour bits from \( R_{23} \).**

We can apply this detour as long as none of the following occurs: (i) \( d = 1 \) or \( w = 1 \); and (ii) \( c = 1 \) and \( d = 4 \), or \( v = 1 \) and \( w = 4 \). We apply the DS as follows:
\[ R_{23} \rightarrow R_{23} - \lambda, \quad R_{21} \rightarrow R_{21} + \lambda, \quad \text{and} \quad R_{15} \rightarrow R_{13} + \lambda \]

**C.1.3. Detour bits from \( R_{31} \).**

We can apply this detour as long as none of the following occurs: (i) \( d = 2 \) or \( w = 2 \); and (ii) \( c = 2 \) and \( d = 4 \), or \( v = 2 \) and \( w = 4 \). We apply the DS as follows:
\[ R_{31} \rightarrow R_{31} - \lambda, \quad R_{32} \rightarrow R_{32} + \lambda, \quad \text{and} \quad R_{21} \rightarrow R_{21} + \lambda \]

By checking the conditions in Theorem 2, for each case, taking into account (C2) and (C3), we can verify that all of them are satisfied. Therefore, regardless of the ordering of the channel gains, there will be at least one detour that we can apply.

**Remark.** If the MGC is in the form of (29) or (30), the proof will follow the same steps, but we will compare the MGC with the conditions in Theorem 2 that are restricted to \( n_{AR} \), that is, (17), and \( n_{BR} \), that is, (24), respectively.

**C.2. Detour Scheme 2**

If the MGC is in the form of
\[ R_{ij} + R_{jk} + R_{kl} + R_{li} + \max(R_{ij}, R_{lj}) + \max(R_{ik}, R_{ik}) \leq n^* \tag{C4} \]

the detoured bits will be subtracted from specific rates selected from the two three-node cycles according to the order of the channel gains in both UL and DL phases. For simplicity, we assume the MGC is in the form of (C4) for \( \{i, j, k, l\} = \{1, 2, 3, 4\} \) and \( \max(R_{13}, R_{11}) = R_{13} \) and \( \max(R_{23}, R_{32}) = R_{23} \), then the MGC is expressed as
\[ R_{12} + R_{23} + R_{34} + R_{41} + R_{13} + R_{42} = n^* + \lambda \tag{C5} \]

We can observe that the MGC contains the following two three-node cycles
\[ R_{34} \rightarrow R_{42} \rightarrow R_{23}, \quad R_{34} \rightarrow R_{41} \rightarrow R_{13} \]

However, by combining conditions (18)–(21) with (25)–(28), for any order of the channel gains, we can obtain
\[ R_{34} + R_{41} + R_{42} + R_{12} + R_{13} + R_{23} \leq n^* \tag{C6} \]

By comparing this condition with the MGC, we get \( R_{34} \geq R_{43} + \lambda \). From the extra SOS conditions, we have
\[ R_{34} + R_{42} + R_{23} + R_{12} + R_{13} + R_{14} = n^* + \beta_1 \tag{C7} \]

By subtracting this condition from the MGC, we have \( R_{41} = R_{14} + \lambda - \beta_1 \). Also, from the upper bound conditions in Theorem 2, we have
\[ R_{12} + R_{13} + R_{14} + R_{23} + R_{24} + R_{34} \leq n^* \]

and by comparing with (C7), we get \( R_{42} \geq R_{24} + \beta_1 \). Again, from the extra SOS conditions in Lemma 1, we have
\[ R_{34} + R_{41} + R_{13} + R_{12} + R_{42} + R_{32} = n^* + \gamma_1 \tag{C8} \]

By subtracting this condition from the MGC, we can get \( R_{23} = R_{32} + \lambda - \gamma_1 \). Also, from the upper bound conditions in Theorem 2, we can get
\[ R_{31} + R_{32} + R_{34} + R_{41} + R_{42} + R_{12} \leq n^* \]

and by comparing with (C8), we obtain \( R_{13} \geq R_{31} + \gamma_1 \), where \( \lambda = \beta_1 + \gamma_1 \).

Now, we need to choose the rates from which we will detour the \( \lambda \)-bits, we have the following options.

**C.2.1. Detour \( \lambda \) bits from \( R_{34} \).**

We can apply this detour as long as none of the following occurs: (i) \( d = 1 \) or \( w = 1 \); and (ii) \( d = 2 \) or \( w = 2 \). We apply the DS as follows:
\[ R_{34} \rightarrow R_{34} - \lambda, \quad R_{31} \rightarrow R_{31} + \gamma_1, \quad R_{14} \rightarrow R_{14} + \gamma_1, \quad R_{32} \rightarrow R_{32} + \beta_1, \quad \text{and} \quad R_{24} \rightarrow R_{24} + \beta_1 \]
C.2.2. Detour $\gamma_1$ bits from $R_{34}$ and $\beta_1$ bits from $R_{42}$.

We can apply this detour as long as none of the following occurs: (i) $d = 1$ or $w = 1$; (ii) $d = 3$ or $w = 3$; and (iii) $c = 1$ and $d = 2$, or $v = 1$ and $w = 2$. We apply the DS as follows:

$$R_{34} \rightarrow R_{34} - \gamma_1, R_{31} \rightarrow R_{31} + \gamma_1, R_{14} \rightarrow R_{14} + \gamma_1,$$

$$R_{42} \rightarrow R_{42} - \beta_1, R_{43} \rightarrow R_{43} + \beta_1, \text{ and } R_{32} \rightarrow R_{32} - \beta_1.$$

C.2.3. Detour $\gamma_1$ bits from $R_{13}$ and $\beta_1$ bits from $R_{42}$.

We can apply this detour as long as none of the following occurs: (i) $d = 3$ or $w = 3$; (ii) $d = 4$ or $w = 4$; (iii) $c = 3$ and $d = 1$, or $v = 3$ and $w = 1$; and (iv) $c = 4$ and $d = 2$, or $v = 4$ and $w = 2$. We apply the DS as follows:

$$R_{13} \rightarrow R_{13} - \gamma_1, R_{14} \rightarrow R_{14} + \gamma_1, R_{43} \rightarrow R_{43} + \gamma_1,$$

$$R_{42} \rightarrow R_{42} - \beta_1, R_{43} \rightarrow R_{43} + \beta_1, \text{ and } R_{32} \rightarrow R_{32} + \beta_1.$$

C.2.4. Detour $\gamma_1$ bits from $R_{34}$ and $\beta_1$ bits from $R_{23}$.

We can apply this detour as long as none of the following occurs: (i) $d = 1$ or $w = 1$; (ii) $d = 4$ or $w = 4$; and (iii) $c = 1$ and $d = 2$, or $v = 1$ and $w = 2$. We apply the DS as follows:

$$R_{34} \rightarrow R_{34} - \gamma_1, R_{31} \rightarrow R_{31} + \gamma_1, R_{14} \rightarrow R_{14} + \gamma_1,$$

$$R_{23} \rightarrow R_{23} - \beta_1, R_{24} \rightarrow R_{24} + \beta_1, \text{ and } R_{43} \rightarrow R_{43} + \beta_1.$$

C.2.5. Detour $\gamma_1$ bits from $R_{41}$ and $\beta_1$ bits from $R_{34}$.

We can apply this detour as long as none of the following occurs: (i) $d = 2$ or $w = 2$; (ii) $d = 3$ or $w = 3$; and (iii) $c = 2$ and $d = 1$, or $v = 2$ and $w = 1$. We apply the DS as follows:

$$R_{41} \rightarrow R_{41} - \gamma_1, R_{43} \rightarrow R_{43} + \gamma_1, R_{31} \rightarrow R_{31} + \gamma_1,$$

$$R_{34} \rightarrow R_{34} - \beta_1, R_{32} \rightarrow R_{32} + \beta_1, \text{ and } R_{24} \rightarrow R_{24} + \beta_1.$$

C.2.6. Detour $\gamma_1$ bits from $R_{41}$ and $\beta_1$ bits from $R_{42}$.

We can apply this detour as long as none of the following occurs: (i) $d = 3$ or $w = 3$; (ii) $c = 3$ and $d = 1$, or $v = 3$ and $w = 1$; and (iii) $c = 3$ and $d = 2$, or $v = 3$ and $w = 2$. We apply the DS as follows:

$$R_{41} \rightarrow R_{41} - \gamma_1, R_{43} \rightarrow R_{43} + \gamma_1, R_{31} \rightarrow R_{31} + \gamma_1,$$

$$R_{42} \rightarrow R_{42} - \beta_1, R_{43} \rightarrow R_{43} + \beta_1, \text{ and } R_{32} \rightarrow R_{32} + \beta_1.$$

C.2.7. Detour $\gamma_1$ bits from $R_{41}$ and $\beta_1$ bits from $R_{23}$.

We can apply this detour as long as none of the following occurs: (i) $d = 3$ or $w = 3$; (ii) $d = 4$ or $w = 4$; and (iii) $c = 4$ and $d = 1$, or $v = 4$ and $w = 1$; and (iv) $c = 3$ and $d = 2$, or $v = 3$ and $w = 2$. We apply the DS as follows:

$$R_{41} \rightarrow R_{41} - \gamma_1, R_{43} \rightarrow R_{43} + \gamma_1, R_{31} \rightarrow R_{31} + \gamma_1,$$

$$R_{23} \rightarrow R_{23} - \beta_1, R_{24} \rightarrow R_{24} + \beta_1, \text{ and } R_{43} \rightarrow R_{43} + \beta_1.$$

C.2.8. Detour $\gamma_1$ bits from $R_{13}$ and $\beta_1$ bits from $R_{34}$.

We can apply this detour as long as none of the following occurs: (i) $d = 2$ or $w = 2$; (ii) $d = 4$ or $w = 4$; and (iii) $c = 2$ and $d = 1$, or $v = 2$ and $w = 1$. We apply the DS as follows:

$$R_{13} \rightarrow R_{13} - \gamma_1, R_{14} \rightarrow R_{14} + \gamma_1, R_{34} \rightarrow R_{34} + \gamma_1,$$

$$R_{34} \rightarrow R_{34} - \beta_1, R_{32} \rightarrow R_{32} + \beta_1, \text{ and } R_{24} \rightarrow R_{24} + \beta_1.$$

C.2.9. Detour $\gamma_1$ bits from $R_{13}$ and $\beta_1$ bits from $R_{23}$.

We can apply this detour as long as none of the following occurs: (i) $d = 4$ or $w = 4$; (ii) $c = 4$ and $d = 1$, or $v = 4$ and $w = 1$; and (iii) $c = 4$ and $d = 2$, or $v = 4$ and $w = 2$. We apply the DS as follows:

$$R_{13} \rightarrow R_{13} - \gamma_1, R_{14} \rightarrow R_{14} + \gamma_1, R_{43} \rightarrow R_{43} + \gamma_1,$$

$$R_{23} \rightarrow R_{23} - \beta_1, R_{24} \rightarrow R_{24} + \beta_1, \text{ and } R_{43} \rightarrow R_{43} + \beta_1.$$

By checking the conditions in Theorem 2, for each case, we can verify that all of them are satisfied. Therefore, regardless the ordering of the channel gains, there will be at least one detour we can apply.

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The capacity of deterministic relay networks with relay messages

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